GLOBAL IN TIME EXISTENCE OF THE STRONG SOLUTION OF THE COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH FREE UPPER SURFACE

BUM JA JIN
MOKPO NATIONAL UNIVERSITY
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Contents

Compressible model of the free boundary value problem

$$\partial_t\rho + \mathrm{div}(\rho\mathbf{u}) = 0 \text{ in } \Omega_t, t \in (0,T),$$
$$\rho(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla)\mathbf{u} - \mathrm{div}\mathbb{T} = -g\rho\mathbf{e}_3, \text{ in } \Omega_t, t \in (0,T),$$

where $-g\mathbf{e}_3$ is the gravity,

$$\mathbb{T} = T(\rho,\mathbf{u}) = -p\mathbb{I} + \Phi,\ \Phi = \lambda\mathrm{div}\mathbf{u}\mathbb{I} + \mu\mathbb{S}, S_{ij} = \partial_{x_i}u_j + \partial_{x_j}u_i,$$

for the isotheral gas $p = k\rho$, $k > 0$, Ω_t is a periodic basin with free upper surface with

$$\Omega = \{(x_1,x_2,x_3) : x_* = (x_1,x_2) \in \mathbb{T}^2,\ 0 < x_3 < \zeta(x_*,t)\}.$$

Let $\Sigma = \mathbb{T}^2 \times \{0\}$ be the bottom of the domain, and $\Gamma_t = \{(x_*,\zeta(x_*,t)) : x_* \in \mathbb{T}^2\}$ be the free upper surface.

Kinematic conditions, that is,

$$\mathbf{u}|_\Sigma = \mathbf{0}\text{(no slip bpundary condition due to the viscosity)},$$
$$\partial_t\zeta = u_3 - u_1\partial_{x_1}\zeta - u_2\partial_{x_2}\zeta (= \sqrt{1+|\nabla_{x*}\zeta|^2}\mathbf{u}\cdot\mathbf{n}) \text{ on } \Gamma_t.$$

By the Capillary theory the shape of the free surface is formed by the difference of the stress tensors from the inside of the fluid and the outside of the fluid, that is,

$$\mathbb{T}(\mathbf{u},p) - (-p_0\mathbf{n}) = \alpha\Delta_{\Gamma_t}(t)\mathbf{x},$$

where p_0 is the atmospheric pressure, $\alpha > 0$ is the surface tension, $\Delta_{\Gamma_t}(t)\mathbf{x}$ is the double mean curvature of the surface Γ_t,

$$\Delta_{\Gamma_t}(t) = g^{-\frac{1}{2}}\sum_{\gamma,\delta=1}^{2}\frac{\partial}{\partial s_\gamma}(g^{\frac{1}{2}}g^{\gamma\delta}\frac{\partial}{\partial s_\delta}), g = \det(g_{\gamma\delta}),$$

$$g_{\gamma\delta} = \frac{\partial\mathbf{x}}{\partial s_\gamma}\cdot\frac{\partial\mathbf{x}}{\partial s_\delta}, (g^{\gamma\delta}) \text{ is the inverse matrix to } (g_{\gamma\delta}),$$

if Γ_t is determined by $\mathbf{x} = \mathbf{x}(s_1,s_2,t), (s_1,s_2) \in \mathbb{T}^2$,

When $\Gamma_t = \{(x_*,\zeta(x_*,t)) : x_* \in \mathbb{T}^2\}$, $\Delta_{\Gamma_t}(t)\mathbf{x} = \mathcal{H}\mathbf{n}$, where

$$\mathcal{H} = \sum_{i=1}^{2}\partial_{x_i}(\frac{\partial_{x_i}\zeta}{\sqrt{1+|\nabla_{x*}\zeta|^2}}) = \nabla_{x*}\cdot(\frac{\nabla_{x*}\zeta}{\sqrt{1+|\nabla_{x*}\zeta|^2}}).$$

Here $\mathbf{u}, \rho, \zeta$ are unknown.

Classical result for the local in time existence

Theorem 0.1 (In anisotropic Sobolev-Slobodetskii space)**.** *Let* $\Gamma_0 \in W_2^{\frac{5}{2}+l}, \in (\frac{1}{2}, 1)$*,* $\rho_0 \in W_2^{l+1},\ \rho_0 \geq R_0 > 0, \alpha > 0$*,* $\mathbf{u}_0 \in W_2^{l+1}(\Omega)$*, and let the following compatibility condition holds:*

$$T(\mathbf{u}_0, p(\rho_0))\mathbf{n}_0 - (-p_0\mathbf{n}_0) = \alpha\mathcal{H}(\zeta_0)\mathbf{n}_0 \text{ on } \Gamma_0.$$

Then there is a unique solution $\mathbf{u} \in W_2^{2+l,1+\frac{l}{2}}$ *on finite time interval* $(0, T_0)$ *whose magnitude* T *depends on the data.*

Lagrangian coordinates representation:

Let us consider a particle trajectory $\mathbf{X}(\xi, t) : \Omega_0 \to \Omega_t$ at the point ξ:

$$\frac{d}{dt}\mathbf{X} = \mathbf{u}(\mathbf{X}, t),\ \mathbf{X}(\xi, 0) = \xi.$$

Denote $\hat{\mathbf{u}} = \mathbf{u}(\mathbf{X}(\xi, t), t),\ \hat{\rho} = \rho(\mathbf{X}(\xi, t), t), \hat{\zeta} = \zeta(\mathbf{X}(\xi, t), t), \hat{p} = p(\hat{\rho})$.

Then $\mathbf{X} = \xi + \int_0^t \hat{\mathbf{u}}(\xi, s)ds := X_{\hat{u}}$.

Denote $\partial_j = \frac{\partial}{\partial \xi_j}$, $b_{jk} = \frac{\partial x_k}{\partial \xi_j} := b_{jk}(\hat{\mathbf{u}})$. Denote $a_{ij} = \frac{\partial \xi_j}{\partial x_i}$,, then

$$a_{ij}b_{jk} = \delta_{ik}.$$

Denote

$$\mathrm{div}_v\mathbf{w} = a_{kl}(\mathbf{v})\partial_l w_k,\ \nabla_v = a_{ij}(\mathbf{v})\partial_j,\ \mathbb{D}_v(\mathbf{w}) = (a_{im}(\mathbf{v})\partial_m w_j + a_{jm}(\mathbf{v})\partial_m w_i),$$

$$\mathbb{S}_v(\mathbf{w}) = \mu\mathbb{D}_v(\mathbf{w}) + \lambda\mathrm{div}_v\mathbf{w}\mathbb{I}, \mathbb{T}_v(\mathbf{w}, q) = \mathbb{S}_v(\mathbf{w}) - q\mathbb{I}.$$

Let $J = \mathrm{det}B$. Direct computation shows that

$$\frac{d}{dt}J = J\mathrm{div}_{\hat{u}}\hat{\mathbf{u}},\ J(\xi, 0) = 1 \Rightarrow J(\xi, t) = e^{(\int_0^t \mathrm{div}_{\hat{u}}\hat{\mathbf{u}})(\xi, s)ds)} := J_{\hat{u}}.$$

Transformation to PDE in the fixed domain

Conservation of the mass transforms to the followings:

$$\partial_t\hat{\rho} + \hat{\rho}\mathrm{div}_{\hat{u}}\hat{\mathbf{u}} = 0 \text{ in } \Omega_0 \times (0,T),$$
$$\hat{\rho}(\xi,0) = \rho_0(\xi) \text{ in } \Omega_0$$

$\Rightarrow \hat{\rho}(\xi,t) = \rho_0(\xi)J_{\hat{u}}^{-1} := \hat{\rho}_{\hat{u}}$.

Kinematic condition of the free upper surface transforms to the followings:

$$\partial_t\hat{\zeta} = \hat{u}_3 \text{ on } \Gamma_0, \hat{\zeta}(\xi,0) = \zeta_0.$$

$\Rightarrow \hat{\zeta}(\xi,t) = \zeta_0(\xi) + \int_0^t \hat{u}_3(\xi_*, \eta_0(\xi_*), s)ds := \hat{\zeta}_{\hat{u}}$.

Let $\hat{\chi} := X_3 - \hat{\zeta}$ and $\chi_0 = \xi_3 - \eta_0(\xi_*) = 0$. Then $\hat{\chi}(\xi,t) = \chi_0 = 0$ for all $t > 0$.

$\Rightarrow \mathbf{n}_{\hat{u}} = \frac{\nabla_{\hat{u}}\chi_0}{\nabla_{\hat{u}}\chi_0}$.

Momentum equation transforms to the followings.

$$\rho_0\partial_t\hat{\mathbf{u}} - J_{\hat{u}}\mathrm{div}\mathbb{S}_{\hat{u}}(\hat{\mathbf{u}}) = \rho_0\hat{\mathbf{f}} - J_{\hat{u}}\nabla_{\hat{u}}p(\rho_0 J_{\hat{u}}^{-1}) \text{ in } \Omega_0,$$
$$\mathbb{S}_{\hat{u}}(\hat{\mathbf{u}})\mathbf{n}_{\hat{u}} + p_0\mathbf{n}_{\hat{u}} = \alpha\Delta_{\Gamma_{\hat{u}}}(t)\mathbf{X}_{\hat{u}} + p(\rho_0 J_{\hat{u}}^{-1})\mathbf{n}_{\hat{u}} \text{ on } \Gamma_0,$$
$$\hat{\mathbf{u}}_{\hat{u}} = 0 \text{ on } \Sigma,\ \hat{\mathbf{u}}(\xi,0) = \mathbf{u}_0(\xi).$$

Here

$$\Delta_{\Gamma_{\hat{u}}}(t)\mathbf{X}_{\hat{u}} = g_{\hat{u}}^{-\frac{1}{2}}\sum_{\gamma,\delta=1}^{2}\frac{\partial}{\partial s_\gamma}(g_{\hat{u}}^{\frac{1}{2}}g_{\hat{u}}^{\gamma\delta}\frac{\partial}{\partial s_\delta}), g_{\hat{u}} = \det(g_{\hat{u},\gamma\delta}),$$
$$g_{\hat{u},\gamma\delta} = \frac{\partial\mathbf{X}_{\hat{u}}}{\partial s_\gamma}\cdot\frac{\partial\mathbf{X}_{\hat{u}}}{\partial s_\delta}, (g_{\hat{u}}^{\gamma\delta}) \text{ is the inverse matrix to } (g_{\hat{u},\gamma\delta}).$$

Solve the nonlinear PDE in the fixed domain Ω_0 with Slip Boundary condition by applying fixed point theorem after linearlization

To resume the solution of the Eulerian system, the Lagrangian transformation $\mathbf{X}_{\hat{u}}$ should be invertible. Since

$$\nabla_\xi\mathbf{X}_{\hat{u}} = \mathbb{I} + \int_0^t \nabla_\xi\hat{\mathbf{u}}(\xi,s)ds,$$

the term $\int_0^t \nabla_\xi\hat{\mathbf{u}}(\xi,s)ds$ should be small, and this is related to the size of T.

So, existence could be obtained for small time interval.

Observation

Initial conditions are given by

$$\mathbf{u}(x,0)=\mathbf{u}_0(x),\ \rho(x,0)=\rho_0(x),\ \zeta(x,0)=\zeta_0(x).$$

Let $\Omega_0=\{(x_*,x_3): x_*\in\mathbb{T}^2,\ 0<x_3<\zeta_0(x_*)\}$.

Let $\int_{\Omega_0}\rho_0 dx = M>0$ and $\int_\Sigma \zeta_0(x_*)dx_* = h_b$.
From the conservation of the mass,

$$\int_{\Omega_t}\rho dx = M,$$

from the incompressibility,

$$|\Omega_t|=|\Omega_0|,\ \text{that is,}\int_\Sigma \zeta(x_*,t)dx_* = \int_\Sigma \zeta_0(x_*)dx_*.$$

Rest state

Let $(\rho, \mathbf{u}, \zeta)$ be the solution of the rest state(equilibrium state), then

$$\mathrm{div}(\rho\mathbf{u}) = 0 \text{ in } \Omega,$$
$$\rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mathrm{div}\mathbb{T} = -g\rho\mathbf{e}_3, \text{ in } \Omega,$$

with the boundary condition

$$\mathbf{u}|_\Sigma = \mathbf{0},$$
$$\mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma.$$
$$\mathbb{T}(\mathbf{u}, p) - (-p_0\mathbf{n}) = \alpha\mathcal{H}\mathbf{n},$$

where

$$\Omega = \{(x_*, x_3) : x_* \in \mathbb{T}^2,\ 0 < x_3 < \zeta(x_*)\},$$

$\Gamma = \left\{(x_*, \zeta(x_*) = h_b) : x_* \in \mathbb{T}^2\right\}$.

Additionally, it should hold that $\int_{\Omega_0} \rho dx = M > 0$ and $\int_\Sigma \zeta dx_* = h_b$.

Taking inner product $\mathbf{u}$ to the momentum equation, then we have the identity

$$\int_\Omega \mu|\mathbb{D}|^2 + \lambda(\mathrm{div}\mathbf{u})^2 dx = 0.$$

This implies that $\mathbf{u} = const$.

$\mathbf{u}|_{\Sigma=0} \Rightarrow \mathbf{u} = \mathbf{0}(:= \mathbf{u}_b)$.

Let $\zeta = h_b(:= \zeta_b)$.

$\mathbf{u} = \mathbf{0} \Rightarrow k\nabla\rho = -g\mathbf{e}_3 \Rightarrow \rho = \rho_* e^{-\frac{g}{k}x_3}(:= \rho_b)$ for some $\rho_* > 0$.

$\zeta = h_b \Rightarrow k\rho(h_b) = p_0 \Rightarrow \rho_* = \frac{p_0}{k}e^{\frac{g}{k}h_b}$.

$\int_{\Omega_0} \rho dx = M \Rightarrow h_b = \frac{k}{g}\ln\left(1 + \frac{Mg}{p_0|\mathbb{T}^2|}\right)$.

Perturbed Equation near the rest state

Let $\sigma = \rho - \rho_b, \eta = \zeta - h_b$, then the FBVP of CNSE are rewritten as follows.

$$\begin{aligned} &\left.\begin{array}{l} \partial_t \sigma + \mathrm{div}(\rho_b \mathbf{u}) + \mathrm{div}(\mathbf{u}\sigma) = 0, \\ (\sigma + \rho_b)(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) - \mathrm{div}\mathbb{T} = -g(\sigma+\rho_b)\mathbf{e}_3 \end{array}\right. \text{ in } \Omega_t, \\ &\mathbf{u}|_\Sigma = \mathbf{0}, \\ &\partial_t \eta = u_3 - (\partial_{x_1}\eta)u_1 - (\partial_{x_2}\eta)u_2 \text{ on } \Gamma_t, \\ &\mathbb{T}\mathbf{n} = (-p_0 + \alpha\mathcal{H})\mathbf{n} \text{ on } \Gamma_t, \\ &\mathbf{u}(x,0) = \mathbf{u}_0,\ \sigma(x,0) = \sigma_0, \eta(x,0) = \eta_0, \end{aligned}$$

where

$$\Omega_t = \{(x_*, x_3) : x_* \in \mathbb{T}^2, x_3 \in (0, \eta + h_b)\},$$
$$\Gamma_t = \{(x_*, \eta(x_*, t) + h_b) : x_8 \in \mathbb{T}^2\}.$$

Additionally,

$$\int_{\Omega_0} \sigma_0 dx = 0,\ \int_\Sigma \zeta_0 dx = 0.$$

Global in time existence and the exponential decay to the rest state

Based on the local existence of the strong solution in some finite time interval, we can derive a priori estimates in the time interval $[0, T]$ to conclude that the maximal time T could be extended to $T = \infty$.

Theorem 0.2. *There is a small number $\epsilon > 0$ so that if*

$$\|\rho_0 - \rho_b\|_{H^2} + \|\mathbf{u}_0\|_{H^2} + \|\zeta_0 - \zeta_b\|_{H^3} < \epsilon,$$

then the solution exists globally in time and

$$(\|\rho(t) - \rho_b\|_{H^2} + \|\mathbf{u}(t)\|_{H^2} + \|\zeta(t) - \zeta_b\|_{H^3})^2 \leq e^{-bt}(\|\rho_0 - \rho_b\|_{H^2} + \|\mathbf{u}_0\|_{H^2} + \|\zeta_0 - \zeta_b\|_{H^3})^2$$

for some positive constant $b > 0$.

We show the theorem by deriving estimates of the solution and its derivatives in Hilbert spaces, purely via energy estimates.

We can adopt Galerkin method to show local in time existence of some strong solution

$$\sigma \in C([0, T); H^2), \eta \in C(0, T; H^3), \mathbf{u} \in C([0, T); H^2).$$

Outline of the proof of the global in time solvability

Step 1.

Take inner product $\mathbf{u}$ to the momentum equations, then we have

$$\frac{d}{dt}E(t) + aD(t) \leq 0,$$

where

$$E(t) = \int_\Omega \rho|\mathbf{u}|^2 + k(\rho \ln \rho - \bar{\rho} \ln \bar{\rho} - (\ln \bar{\rho} + 1)(\rho - \hat{\rho}))dx + \int_\Sigma \sqrt{1 + |\nabla_{x*}\zeta|^2} - 1 dx_*$$
$$\sim \|\mathbf{u}\|_{L^2}^2 + \|\sigma\|_{L^2}^2 + \|\nabla_{x*}\zeta\|_{L^2}^2$$

when $\|\sigma\|_{L^\infty} + \|\nabla_{x*}\eta\|_{L^\infty}$ is small enough, and

$$D(t) \sim \|\nabla \mathbf{u}\|_{L^2}^2.$$

Step 2.

Choose $\mathbf{V}$ satisfying that

$$\mathrm{div}(\rho_b \mathbf{\Phi}) = \sigma \text{ in } \Omega,$$
$$\mathbf{V}|_\Sigma = 0,\ \rho_b \mathbf{V} \cdot \mathbf{n} = \frac{1}{n_3}\eta,$$

then

$$\frac{d}{dt}F(t) + a_1 D_0(t) \leq S(\mathcal{E}(t))(D(t) + D_0(t)) + cD,$$

where $0 \leq F(t) \leq cE(t)$, $D_1(t) \sim \|\nabla \eta\|_{L^2}^2 + \|\sigma\|_{L^2}^2$,

$$\mathcal{E}(t) \sim \|\sigma\|_{H^2}^2 + \|\mathbf{u}\|_{H^2}^2 + \|\eta\|_{H^3}^2 + \|\partial_t \sigma\|_{L^2}^2 + \|\partial_t \mathbf{u}\|_{L^2}^2 + \|\partial_t \eta\|_{H^1}^2.$$

Step 3.

$$\frac{d}{dt}(E(t) + \delta F(t)) + (b - c\delta - \delta S(\mathcal{E}(t)))(D(t) + D_0(t)) \leq 0,$$

Here $b = \min\{a, \delta a_1\}$.

Observe that $E(t) + \delta F(t) \sim (D(t) + D_0(t))$ when $\|\sigma\|_{L^\infty} + \|\nabla_{x*}\eta\|$ is small enough. If δ is small enough, then we have exponential decay of $E(t)$.

Therefore it is necessary to derive the estimate of higher order derivatives up to the regularity $\|\sigma\|_{L^\infty} + \|\nabla_{x*}\eta\|$, or up to $\|\sigma\|_{H^2} + \|\nabla_{x*}\eta\|_{H^2}$.

Step 4. Derive energy estimate concerning with the Higher order derivatives, that is,

$$\mathcal{E}(t) \sim \|\sigma\|_{H^2}^2 + \|\mathbf{u}\|_{H^2}^2 + \|\eta\|_{H^3}^2 + \|\partial_t \sigma\|_{L^2}^2 + \|\partial_t \mathbf{u}\|_{L^2}^2 + \|\partial_t \eta\|_{H^1}^2,$$

and

$$\mathcal{D}(t) \sim \|\sigma\|_{H^2}^2 + \|\mathbf{u}\|_{H^3}^2 + \|\partial_t \mathbf{u}\|_{H^1}^2 + \|\eta\|_{H^3}^2.$$

Step 5. Finally, obtain Gronwall inequality

$$\frac{d}{dt}\bar{\mathcal{E}}(t) + \bar{\mathcal{D}}(t) \leq c_2 \bar{\mathcal{D}}(t) S(\bar{\mathcal{E}}), \text{ for all } t \in (0, T),$$

for some $S \in C^1$ which is strictly increasing near the origin, that is, $S'(0) > 0, S(0) = 0$. Here $\bar{\mathcal{E}} \sim \mathcal{E}, \bar{\mathcal{D}} \sim \mathcal{D}$.

Theorem 0.3 (Analytic continuation)**.** *Let X be a Banach space, $y \in C([0,T);X)$.*

If $\lim_{t\to T} y(t) < \infty$, then there is $T_1 > T$ so that $y(t) \in C([0,T_1);X)$.

Lemma 0.4 (Gronwall's lemma)**.** *Suppose the following inequalities hold as long as $\mathcal{Y}(t) < C_0$.*

$$\mathcal{Z}(t) \geq c_1 \mathcal{Y}(t),$$
$$\frac{d}{dt}\mathcal{Y}(t) + \mathcal{Z}(t) \leq c_2 \mathcal{Z}(t) S(\mathcal{Y}),$$

for some function $S \in C_1$ which is strictly increaing near the origin, that is, $S'(0) > 0, S(0) = 0$. Here C_0, c_1 and c_2 are positive constants independent of T. Then there is a small positive constants κ_0, which depends only on C_0 and c_2, such that if $\mathcal{Y}(0) \leq \frac{\kappa_0}{2}$, then $\mathcal{Y}(t)$ exists globally in time and satisfies the inequality

$$\mathcal{Y}(t) < \mathcal{Y}(0) e^{-\frac{c_1}{2}t}, \text{ for all } t > 0.$$

How to derive Higher order derivatives

i)*Time derivative*

Multiply $\partial_t(\cdot)$ to the equations differentiated by t, and hen integrate over Ω.

$$\Rightarrow \frac{d}{dt}E_2 + D_2 \leq c\sqrt{\mathcal{E}}\mathcal{D},$$

where $E_2 \sim \|\partial_t\sigma\|_{L^2}^2 + \|\partial_t\mathbf{u}\|_{L^2}^2 + \|\nabla_{x_*}\partial_t\eta\|_{L^2}^2$ and $D_2 \sim \|\nabla\partial_t\mathbf{u}\|_{L^2}^2$.

ii)*First order tangential derivatives*

If $F(x,t) = 0$ on $x_3 = 0$, then $\partial_{x_i}F(x,t) = 0$ $x_3 = 0$.

On the other hand, if $F(x,t) = 0$ on Γ_t, that is, if $F(x_1, x_2, h_b + \eta(x_1, x_2, t)) = 0$, then

$$\partial_{x_1}F + (\partial_{x_1}\eta)\partial_{x_3}F = 0 \text{ on } \Gamma_t.$$

Hence we introduce

$$\tilde{\partial}_i = \partial_{x_i} + (\partial_{x_i}\tilde{\eta})\partial_3,$$

where $\tilde{\eta} \in H^{\frac{7}{2}}(\Omega_t)$ is an extension of $\eta \in H^3(\Gamma_t)$ so that $\tilde{\eta} = 0$ on $x_3 = 0$ and $\tilde{\eta} = \eta$ on Γ_t.

Multiply $\hat{\partial}_i(\cdot)$ to the equations differentiated by $\hat{\partial}_i$, and hen integrate over Ω.

$$\Rightarrow \frac{d}{dt}E_3 + D_3 \leq c\sqrt{\mathcal{E}}\mathcal{D},$$

where $E_2 \sim \|\hat{\partial}_i\sigma\|_{L^2}^2 + \|\hat{\partial}_i\mathbf{u}\|_{L^2}^2 + \|\nabla_{x_*}^2\eta\|_{L^2}^2$ and $D_2 \sim \|\nabla\hat{\partial}_i\mathbf{u}\|_{L^2}^2$.

iii)*Second order tangential derivatives*

Multiply $\hat{\partial}_i\hat{\partial}_j(\cdot)$ to the equations differentiated by $\hat{\partial}_i\hat{\partial}_j$, and hen integrate over Ω.

$$\Rightarrow \frac{d}{dt}E_4 + D_4 \leq c\sqrt{\mathcal{E}}\mathcal{D},$$

where $E_3 \sim \|\hat{\partial}_i\hat{\partial}_j\sigma\|_{L^2}^2 + \|\hat{\partial}_i\hat{\partial}_j\mathbf{u}\|_{L^2}^2 + \|\nabla_{x_*}^3\eta\|_{L^2}^2$ and $D_3 \sim \|\nabla\hat{\partial}_i\hat{\partial}_j\mathbf{u}\|_{L^2}^2$.

iv) Set

$$E^* \sim \|\mathbf{u}\|_{L^2}^2 + \|\nabla_{x*}\mathbf{u}\|_{L^2}^2 + \|\nabla_{x*}^2\mathbf{u}\|_{L^2}^2 + \|\partial_t\mathbf{u}\|_{L^2}^2 + \|\sigma\|_{L^2}^2$$
$$+\|\nabla_{x*}\sigma\|_{L^2}^2 + \|\nabla_{x*}^2\sigma\|_{L^2}^2 + \|\partial_t\sigma\|_{L^2}^2 + \|\zeta\|_{H^3}^2 + \|\partial_t\zeta\|_{H^1}^2,$$

and

$$D^* \sim \|\mathbf{u}\|_{H^1}^2 + \|\nabla\nabla_{x*}\mathbf{u}\|_{L^2}^2 + \|\nabla\nabla_{x*}^2\mathbf{u}\|_{L^2}^2 + \|\partial_t\mathbf{u}\|_{H^1}^2 + \|\sigma\|_{L^2}^2$$
$$+\|\nabla_{x*}\sigma\|_{L^2}^2 + \|\nabla_{x*}^2\sigma\|_{L^2}^2 + \|\zeta\|_{H^3}^2.$$

Combine Step 1, Step 2, i)-iii), then we have

$$\Rightarrow \frac{d}{dt}E^* + D^* \leq c\sqrt{\mathcal{E}}\mathcal{D}.$$

v)*normal derivatives of the velocity*

From momentum equation for the first two components, that is,

$$\mu\partial_3^2\mathbf{u}^* = (\sigma+\rho_b)(\partial_t\mathbf{u}^* + (\mathbf{u}\cdot\nabla)\mathbf{u}^*) - \mu\Delta_*\mathbf{u}^* - (\mu+\lambda)\nabla_*\mathrm{div}\mathbf{u} + k\rho_b\nabla_*\frac{\sigma}{\rho_b},$$

$$\mu\|\partial_3^2\mathbf{u}^*\|_{L^2}^2 \le cD^* + c\sqrt{\mathcal{E}}\mathcal{D}.$$

vi)*normal derivatives of the density*

Combine the momentum equation for the third component and the normal derivative of transport equation, we have

$$(2\mu+\lambda)(\partial_t\partial_3\sigma + (\mathbf{u}\cdot\nabla\partial_3\sigma + (\partial_3\sigma)\mathrm{div}\mathbf{u}) + k\rho_b\partial_3\sigma = \cdots + \mu\rho_b(\Delta u_3 - \partial_3\mathrm{div}\mathbf{u}),$$

$$\Rightarrow \frac{2\mu+\lambda}{2}\frac{d}{dt}\int|\partial_3\sigma|^2 + \int\rho_b|\partial_3\sigma|^2dx \le cD^* + c\sqrt{\mathcal{E}}\mathcal{D}.$$

vii)*Time derivative -1*

Multiply $\partial_t(\cdot)$ to the equations, and hen integrate over Ω.

$$\Rightarrow \frac{d}{dt}E_1 + D_1 \le c\sqrt{\mathcal{E}}\mathcal{D} + cD^*,$$

where $E_1 \sim \|\nabla\mathbf{u}\|_{L^2}^2$ and $D_1 \sim \|\partial_t\mathbf{u}\|_{L^2}^2$.

viii) From v)-vii)

$$\Rightarrow \frac{d}{dt}E_0 + D_0 \leq c\sqrt{\mathcal{E}}\mathcal{D} + cD^*,$$

where $E_0 \sim \|\nabla \mathbf{u}\|_{L^2}^2 + \|\partial_3 \sigma\|_{L^2}^2$ and $D_1 \sim \|\partial_t \mathbf{u}\|_{L^2}^2 + \|\partial_3 \sigma\|_{L^2}^2 + \|\partial_3^2 \mathbf{u}\|_{L^2}^2$.

ix) From iv) and viii)

$$\Rightarrow \frac{d}{dt}(E^* + \theta E_0) + (D^* + \theta D_0) \leq c\sqrt{\mathcal{E}}\mathcal{D}$$

for some small $\theta > 0$.

Observe that

$$\mathcal{E} \sim E^* + \theta E_0$$

and

$$\mathcal{D} \sim D^* + \theta D_0.$$

Weak solution for any large data?

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