SOLVABILITY OF THE STATIONARY COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH LARGE FORCE

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Contents

Incompressible Navier-Stokes Equations

Let us consider incompressible viscous fluid with constant density $\bar{\rho} > 0$ in a bounded domain $\Omega \subset \mathbb{R}^n,\ n = 2, 3.$

$$\begin{aligned} \mathrm{div}\mathbf{U} &= 0, \\ \bar{\rho}(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U}) - \mu \Delta \mathbf{U} + \nabla P &= \bar{\rho}\mathbf{f}, \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{0}, \\ \frac{1}{|\Omega|}\int_\Omega P dx &= 0. \end{aligned}$$

Here $\mathbf{U}$ is the velocity of the fluid,
P is the pressure of the fluid,
$\mu > 0$ is the viscosity of the fluid.

Stationary model

$$\begin{aligned} \mathrm{div}\mathbf{U} &= 0, \\ \bar{\rho}\mathbf{U} \cdot \nabla \mathbf{U} - \mu \Delta \mathbf{U} + \nabla P &= \bar{\rho}\mathbf{f}, \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{0}. \end{aligned}$$

Fixed point theorem for the existence

Let us consider a mapping $T : X \to X$ for some Banach space define by $\tilde{\mathbf{u}} \mapsto \mathbf{u}$, where $\mathbf{u}$ is a solution of the Stokes system

$$\begin{aligned} \mathrm{div}\mathbf{u} &= 0, \\ -\mu\Delta\mathbf{u} + \nabla p = \bar{\rho}\mathbf{f} - \bar{\rho}\tilde{\mathbf{u}} \cdot \nabla\tilde{\mathbf{u}} &:= F(\mathbf{f}, \tilde{\mathbf{u}}), \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{0}. \end{aligned}$$

We would like to show that there is a fixed point $\mathbf{u}$ so that $T(\mathbf{u}) = \mathbf{u}$.

- Contraction mapping theorem
  (for small data; existence and uniqueness of strong solution)
- Galerkin method
  (for large data; uniqueness for small data)
- Leray-Schauder theorem
  (for large data; uniqueness for small data)

Theorem 0.1 (Contraction mapping theorem)**.** *Let T be a mapping $T : X \to X$ on Banach space X, and there is $\theta < 1$ so that*

$$\|T(x) - T(y)\|_X \leq \theta \|x - y\|_X \text{ for all } x, y \in X.$$

Then there is fixed point x so that $T(x) = x$.

Let $\mathbf{u}_0 = \mathbf{0}$ be given, and

$$\mathbf{u}_1 = T(\mathbf{u}_0),\ \mathbf{u}_2 = T(\mathbf{u}_1),\ \cdots, \mathbf{u}_{n+1} = T(\mathbf{u}_n), \cdots,$$

where $T : X = H_0^2 \to X$ is a mapping that $T(\mathbf{u}_n) = \mathbf{u}_{n+1}$, $\mathbf{u}_{n+1}$ is a solution of the Stokes system $\mathbf{u}$ is a solution of the Stokes system

$$\begin{aligned} \mathrm{div}\mathbf{u}_{n+1} &= 0, \\ -\mu\Delta\mathbf{u}_{n+1} + \nabla p_{n+1} &= \bar{\rho}\mathbf{f} - \bar{\rho}\mathbf{u}_n \cdot \nabla\mathbf{u}_n, \\ \mathbf{u}_{n+1}|_{\partial\Omega} &= \mathbf{0}. \end{aligned}$$

By the well known theory for the Stokes system we have

$$\|\mathbf{u}_{n+1}\|_X \leq c\|\mathbf{f}\|_Y + c\|\mathbf{u}_n\|_X^2, \text{ where } Y = L^2.$$

There is a small $M_0 > 0$ so that if $\|\mathbf{f}\|_Y \leq M_0$, then

$$\|\mathbf{u}_{n+1}\|_X \leq 2c\|\mathbf{f}\|_Y.$$

To show the convergence of $\{\mathbf{u}_n : n = 1, 2, \cdots\}$
it is enough to show that

$$\|\mathbf{u}_{n+1} - \mathbf{u}_n\|_X \leq \theta\|\mathbf{u}_{n+1} - \mathbf{u}_n\|_X \text{ for some } \theta < 1$$

and for some Banach space X.

Let $U_n = \mathbf{u}_{n+1} - \mathbf{u}_n$. Then

$$\begin{aligned} \mathrm{div}\mathbf{U}_n &= 0, \\ -\mu\Delta\mathbf{U}_n + \nabla P_n &= -\bar{\rho}\tilde{\mathbf{u}}_n \cdot \nabla\tilde{\mathbf{U}}_{n-1} - \bar{\rho}\tilde{\mathbf{U}}_n \cdot \nabla\tilde{\mathbf{u}}_{n-1}, \\ \mathbf{U}_n|_{\partial\Omega} &= \mathbf{0}. \end{aligned}$$

Let $X = H^k(\Omega),\ k = 1, 2, \cdots$. By variational formulation tested by $\mathbf{U}_n$, we have

$$\|\mathbf{U}_n\|_X \leq c(\|\mathbf{u}_n\|_X + \|\mathbf{u}_{n-1}\|_X)\|\mathbf{U}_{n-1}\|_X.$$

Hence we need smallness of $\theta = c(\|\mathbf{u}_n\|_X + \|\mathbf{u}_{n-1}\|_X)$ which comes from the smallness of $\mathbf{f}$.

Theorem 0.2 (Leray-Schauder's theorem)**.** *Let $\mathcal{B}$ be a Banach space, and let $T : \mathcal{B} \times [0,1] \to \mathcal{B}$ be a compact mapping , there is $x \in \mathcal{B}$ such that $T(x,0) = 0$, there is $M > 0$ so that*

$$\textit{if } T(x,\sigma) = x, \textit{ then } \|x\|_{\mathcal{B}} < M.$$

Then $T(\cdot, 1)$ has a fixed point in $x \in \mathcal{B}$ such that $T(x,1) = x$.

Let $((\cdot,\cdot))$ be the inner product in the Hilbert space $H = H^1_{0,\sigma}(\Omega)$ defined by

$$((\mathbf{u},\mathbf{v})) := \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}dx.$$

Define L_1 by

$$L_1(\mathbf{v}) := \int_\Omega \mathbf{u}\cdot(\mathbf{u}\cdot\nabla\mathbf{v})dx.$$

Then $L_1 : H \to H$ is a bounded linear operator. By Riesz theorem, there is $A\mathbf{u} \in H$ $((A(\mathbf{u}),\mathbf{v})) = \int_\Omega \mathbf{u}\cdot(\mathbf{u}\cdot\nabla\mathbf{v})dx$.

Define L_2 by

$$L_2(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}dx.$$

Then $L_2 : H \to H$ is a bounded linear operator. By Riesz theorem, there is $\mathbf{f} \in H$ $((\mathbf{F},\mathbf{v})) =< \mathbf{f},\mathbf{v} >$. Here $< \cdot,\cdot >$ is a duality paring between H^1_0 and $H^{-1} = (H^1_0)*$.

Hence the variational formulation of the NSE can be rewritten by

$$((\mu\mathbf{u} - A\mathbf{u} - \mathbf{F},\mathbf{v})) = 0 \text{ for all } \mathbf{v} \in X.$$

Let $T\mathbf{u} = \frac{1}{\mu}(A\mathbf{u} + \mathbf{F})$. To find a solution is to find of fixed point of T.

T is a compact operator.

$\because$

Let $\{\mathbf{u}_n : n = 1,2,\cdots\}$ be a bounded sequence in $H = H^1_0(\Omega)$. There is a subsequence $\{\mathbf{u}_{n_k} : k = 1,2,\cdots\}$ converging weakly to some $\mathbf{u} \in H^1_0(\Omega)$.

By Rellich Kondrachov compactness theorem, $H^1_0(\Omega)$ is compactly embedded into $L^4(\Omega)$. Hence $\{\mathbf{u}_{n_k} : k = 1,2,\cdots\}$ converges strongly to some $\mathbf{u} \in L^4(\Omega)$. Since

$$((T(\mathbf{u}_n) - T(\mathbf{u}),\mathbf{v})) = \int_\Omega (\mathbf{u}_n - \mathbf{u})\cdot[\mathbf{u}_n\cdot\nabla\mathbf{v}] + \mathbf{u}\cdot[(\mathbf{u}_n - \mathbf{u})\cdot\nabla\mathbf{v}]dx,$$

We conclude that

$$\|T(\mathbf{u}_{n_k}) - T(\mathbf{u})\|_H \le c\|\mathbf{u}_{n_k} - \mathbf{u}\|_{L^4}(\|\mathbf{u}_{n_k}\|_H + \|\mathbf{u}\|_H) \to 0 \text{ as } k \to \infty.$$

Galerkin method-Brower fixed point theorem

Let $\{\phi_k : k = 1, 2, \cdots\} \subset C^\infty_{0,\sigma}$ be the countable dense subset of $H = H^1_0(\Omega)$ with $((\phi_k, \phi_l)) = \delta_{kl}$.

Let $V_m = \text{span}\{\phi_1, \cdots, \phi_m\}$. Find $\mathbf{u}_m = \sum_{k=1}^m a_{mk}\phi_k \in V_m$ satisfying that

$$\mu((\mathbf{u}_m, \phi_i)) + \int_\Omega \mathbf{u}_m \cdot (\mathbf{u}_m \cdot \nabla\phi_i)dx =< \mathbf{f}, \phi_i >, k = 1, \cdots, m.$$

This is equal to find $\xi = (a_{m1}, \cdots, a_{mm})$ so that

$$\mu a_{mi} + \sum_{j=1}^m a_{mj}a_{ml}M_{jli} = C_i,$$

where $M_{jli} = \int_\Omega \phi_j \cdot (\phi_l \cdot \nabla\phi_i)dx$, $C_i =< \mathbf{f}, \phi_i >$.

Let $P : \mathbb{R}^n \to \mathbb{R}^n$, $[P\xi]_i = \mu a_{mi} + \sum_{j=1}^m a_{mj}a_{ml}M_{jli}- < \mathbf{f}, \phi_i >$. Observe that

$$[\xi, P\xi] = \mu\|\mathbf{u}_m\|^2_H- < \mathbf{f}, \mathbf{u}_m >\geq \|\mathbf{u}_m\|_H(\mu\|\mathbf{u}_m\|_H - \|\mathbf{f}\|_{H'}.$$

Here $[\cdot, \cdot]$ is a scalar product in $\mathbb{R}^n$. Hence $\xi^T P\xi > 0$ if $\|\xi\| = \|\mathbf{u}_m\|_H > \frac{\|\mathbf{f}\|_{H'}}{\mu}$.

By Brower fixed point theorem there is $\xi = (a_{m1}, \cdots, a_{mm}) \in \mathbb{R}^n$ with $\|\xi\| \leq \frac{\|\mathbf{f}\|_{H'}}{\mu}$ and

$$P(\xi) = 0.$$

That is, there is $\mathbf{u}_m = \sum_{k=1}^m a_{mk}\phi_k \in H$ satisfying that

$$\|\xi\| \leq \frac{\|\mathbf{f}\|_{H'}}{\mu},$$

$$\mu((\mathbf{u}_m, \phi_i)) + \int_\Omega \mathbf{u}_m \cdot (\mathbf{u}_m \cdot \nabla\phi_i)dx =< \mathbf{f}, \phi_i >, k = 1, \cdots, m.$$

Since $\{\mathbf{u}_m : m = 1, 2, \cdots\}$ is bounded in $H^1_0(\Omega)$, There is a subsequence $\{\mathbf{u}_{n_k} : k = 1, 2, \cdots\}$ converging weakly to some $\mathbf{u} \in H^1_0(\Omega)$.

By Rellich Kondrachov compactness theorem, $H^1_0(\Omega)$ is compactly embedded into $L^4(\Omega)$. Hence $\{\mathbf{u}_{n_k} : k = 1, 2, \cdots\}$ converges strongly to some $\mathbf{u} \in L^4(\Omega)$.

Fix i. Passing to the limit to the identity

$$\mu((\mathbf{u}_{m_k}, \phi_i)) + \int_\Omega \mathbf{u}_{m_k} \cdot (\mathbf{u}_{m_k} \cdot \nabla\phi_i)dx =< \mathbf{f}, \phi_i >,$$

we have

$$\mu((\mathbf{u}, \phi_i)) + \int_\Omega \mathbf{u} \cdot (\mathbf{u} \cdot \nabla\phi_i)dx =< \mathbf{f}, \phi_i > .$$

Since the above holds for any i, we conclude that

$$\mu((\mathbf{u}, \phi)) + \int_\Omega \mathbf{u} \cdot (\mathbf{u} \cdot \nabla\phi)dx =< \mathbf{f}, \phi > \text{ for any } \phi \in H^1_0(\Omega).$$

Theorem 0.3 (Brower fixed point theorem)**.** *Let* $S : \mathbb{R}^n \to \mathbb{R}^n$ *be a linear operator. If there is* $k > 0$ *such that*

$$\|\xi \leq k \Rightarrow \|S\xi\| \leq k,$$

then there is a fixed point ξ *with* $\|\xi\| \leq k$.

Corollary 0.4. *Let* $P : \mathbb{R}^n \to \mathbb{R}^n$ *be a linear operator. If there is* $k > 0$ *such that*

$$[\xi, P\xi] > 0 \textit{ for } |\xi| = k > 0,$$

then there is zero point ξ *with* $\|\xi\| \leq k$.

Theorem 0.5 (Rellich-Kondrachov compactness theorem)**.** *Let $\Omega \subset \mathbb{R}^n$, Ω_0 be a bounded subdomain of Ω and Ω_0^k be an intersection of Ω_0 with a k dimensional plane in $\mathbb{R}^n$. Let $j \geq 0$ and $m \geq 1$ be integers, and let $1 \leq p < \infty$.*

(1) Ω *If Ω satisfies the cone condition and $mp \leq n$, then the following imbeddings are compact:*

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_0^k) \text{ if } 0 < n - mp < k \leq n \text{ and } 1 \leq q < kp/(n-mp),$$

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_0^k) \text{ if } n = mp, 1 \leq k \leq n \text{ and } 1 \leq q < \infty.$$

(2) *If Ω satisfies the cone condition and $mp > n$, then the followings are compact:*

$$W^{j+m,p}(\Omega) \to C_B^j(\Omega_0),$$

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_0^j) \text{ if } 1 \leq q < \infty.$$

(3) *If Ω satisfies the strong local Lipschitz condition, then the following imbeddings are compact:*

$$W^{j+m,p}(\Omega) \to C^j(\overline{\Omega_0}) \text{ if } mp > n,$$

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega_0}) \text{ if } mp > n \geq (m-1)p \text{ and } 0 < \lambda < m - \frac{n}{p}.$$

(4) *If Ω is an arbitrary domain in $\mathbb{R}^n$, the imbeddings $(1)-(3)$ are compact provided $W^{j+m,p}(\Omega)$ is replaced by $W_0^{j+m,p}(\Omega)$.*

Compressible Navier-Stokes Equations

Let incompressible viscous fluid with constant density $\bar{\rho} > 0$ in a bounded domain $\Omega \subset \mathbb{R}^n,\ n = 2, 3.$

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) 0, \\ \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \mu \Delta \mathbf{u} - \lambda \nabla \operatorname{div} \mathbf{u} + \nabla p = \rho \mathbf{f}, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}. \end{aligned}$$

Here $\mathbf{u}$ is the velocity of the fluid,
$p = R\rho^\gamma, \gamma \geq 1$ is the pressure of the fluid,
$\mu > 0$ and λ are the bulk viscosity and shear viscosity, respectively, of the fluid.

Stationary model

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \rho \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} - \lambda \nabla \operatorname{div} \mathbf{u} + \nabla p = \rho \mathbf{f}, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}. \end{aligned}$$

Scaling to dimensionless form

$t_*, l_*, u_* = \frac{l_*}{t_*}, \rho_*, f_*$: Characteristic quantity of $t, \mathbf{x}, \mathbf{u}, \rho, \mathbf{f}$.

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0,$$
$$\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \frac{1}{Re_1} \Delta \mathbf{u} - \frac{1}{Re_2} \nabla \mathrm{div} \mathbf{u} + \frac{1}{Ma^2} \nabla p = \frac{1}{Fr^2} \rho \mathbf{f},$$
$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}.$$

Here $Re_1 = \frac{\rho_* l_* u_*}{\mu} := \frac{1}{\mu_1}, Re_2 = Re_1 = \frac{\rho_* l_* u_*}{\lambda} := \frac{1}{\lambda_1}$ are Reynolds numbers, $Ma^2 = \frac{\rho_* l_* t}{u_* t_* p(\rho_*)} \sim \frac{u_*^2}{p'(\rho_*)} := \epsilon^2$ are Mach numbers, $Fr^2 = \frac{t_*}{u_* f_*}$.

Low Mach number limit to the incompressible fluid

Consider the case that μ_1, μ_2, Fr are fixed but ϵ is going to small, which is the case $u_* \sim \epsilon, \mu \sim \epsilon, \lambda \sim \epsilon, t_* \sim \frac{1}{\epsilon}, f_* \sim \epsilon^2, l_* \sim 1, \rho_* \sim 1$.

Formally, assuming

$$\rho = \bar{\rho} + \Pi \epsilon^2 + O(\epsilon^3),$$

$$\mathrm{div}(\bar{\rho} \mathbf{u}) = O(\epsilon^2),$$
$$\bar{\rho}(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \frac{1}{Re_1} \Delta \mathbf{u} - \frac{1}{Re_2} \nabla \mathrm{div} \mathbf{u} + \nabla \Pi = \frac{1}{Fr^2} \rho \mathbf{f} + O(\epsilon),$$
$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}.$$

Stationary model

$$\begin{aligned} \mathrm{div}(\rho\mathbf{u}) &= 0, \\ \rho\mathbf{u}\cdot\nabla\mathbf{u} - \mu_1\Delta\mathbf{u} - \lambda_1\nabla\mathrm{div}\mathbf{u} + \frac{1}{\epsilon^2}\nabla p &= \rho\mathbf{f}, \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{0}. \end{aligned}$$

• Known Result: Existence of incompressible flow for any large external force (without Uniqueness).

$$\begin{aligned} \mathrm{div}(\bar{\rho}\mathbf{U}) &= 0, \\ \bar{\rho}\mathbf{U}\cdot\nabla\mathbf{U} - \mu_1\Delta\mathbf{U} - \mu_2\nabla\mathrm{div}\mathbf{U} + \nabla P &= \bar{\rho}\mathbf{f}, \\ \mathbf{U}|_{\partial\Omega} &= \mathbf{0}. \end{aligned}$$

Question: Is this the Low Mach number limit of the compressible fluid with $\frac{1}{|\Omega|}\int_\Omega \rho dx = \bar{\rho} > 0$?

Without loss of generality, assume that $p'(\bar{\rho}) = 1$.

Setting

$$\rho = \bar{\rho} + \epsilon^2(\sigma + P), \ \mathbf{u} = \mathbf{U} + \mathbf{v},$$

$$\begin{aligned} \mathrm{div}(\bar{\rho}\mathbf{v}) + \epsilon^2\mathrm{div}(\sigma(\mathbf{U}+\mathbf{v})) &= \epsilon^2 G(\sigma, \mathbf{v}, \mathbf{U}, P), \\ \bar{\rho}(\mathbf{U}+\mathbf{v})\cdot\nabla\mathbf{v} + \bar{\rho}\mathbf{v}\cdot\nabla U - \mu_1\Delta\mathbf{v} - \lambda_1\nabla\mathrm{div}\mathbf{v} + \nabla\sigma &= \epsilon^2\mathbf{F}(\sigma, \mathbf{v}, \mathbf{U}, P), \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{0}, \\ \frac{1}{|\Omega|}\int_\Omega \sigma dx &= 0. \end{aligned}$$

Here

$$\begin{aligned} G &= -2\mathrm{div}(P(\mathbf{U}+\mathbf{v})), \\ \mathbf{F} &= (P+\sigma)\mathbf{f} - (P+\sigma)(\mathbf{U}+\mathbf{v})\cdot\nabla(\mathbf{U}+\mathbf{v}) \\ &+\nabla\Big(\frac{p(\bar{\rho}+\epsilon^2(\sigma+P)) - p(\bar{\rho}) - p'(\bar{\rho})\epsilon^2(\sigma+P)}{\epsilon^2}\Big). \end{aligned}$$

Theorem 0.6 (Schauder's fixed point theorem)**.** *Let $\mathcal{C}$ be closed convex subspace of Banach space $\mathcal{B}$, and let $T : \mathcal{C} \to \mathcal{C}$ be a continuous mapping, that is,*

$$\text{for each } \epsilon > 0, \text{ there is } \delta > 0 \text{ such that } \|x - y\|_{\mathcal{B}} < \delta \Rightarrow \|T(x) - T(y)\|_{\mathcal{B}} < \epsilon,$$

and $T(\mathcal{C})$ is precompact, that is,

$$\{x_n : n = 1, 2, \cdots\} \text{ is bounded sequence, then there is subsequence } \{x_{n_k} : k = 1, 2, \cdots\}$$
$$\text{so that } T(x_{n_k}) \text{ converges strongly to some } T(x) \text{ for some } x \in \mathcal{C}.$$

Then T has a fixed point in $\mathcal{C}$ such that $Tx = x$.

Decomposition, and linearlization

Observation

Introduce an operator $T : X \to X,\ (\tilde{\sigma}, \tilde{\mathbf{v}}) \mapsto (\sigma, \mathbf{v})$ defined by that for each given data $(\tilde{\sigma}, \tilde{\mathbf{v}})$, let $(\sigma, \mathbf{v})$ be the solution of

$$\begin{aligned} (\sigma - \tilde{\sigma}) + \mathrm{div}(\bar{\rho}\mathbf{v}) + \epsilon^2 \mathrm{div}(\sigma(\mathbf{U} + \mathbf{v})) &= \epsilon^2 G(\tilde{\sigma}, \tilde{\mathbf{v}}, \mathbf{U}, P), \\ \bar{\rho}(\mathbf{U} + \mathbf{v}) \cdot \nabla \mathbf{v} - \mu_1 \Delta \mathbf{v} - \lambda_1 \nabla \mathrm{div} \mathbf{v} + \nabla \sigma &= \epsilon^2 \mathbf{F}(\tilde{\sigma}, \tilde{\mathbf{v}}, \mathbf{U}, P), \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{0}. \end{aligned}$$

Let $X = \{(\sigma, \mathbf{v}) \in H_0^3 \times H^2 : \|\sigma\|_{H^2} + \|\mathbf{v}\|_{H^3} \leq M\}$. To apply Schauder theorem, we should find $M > 0$ so that if $(\tilde{\sigma}, \tilde{\mathbf{u}}) \in X$, then $T(\tilde{\sigma}, \tilde{\mathbf{u}})) = (\sigma, \mathbf{v}) \in X$.

According to the "A Priori estimate", it does not seem to be possible to have such M if $\mathbf{U}$ is not small enough, in other words, if $\mathbf{f}$ is not small enough.

Decomposition

$$
\begin{aligned}
\operatorname{div}(\bar{\rho}\mathbf{U}) &= 0,\\
\bar{\rho}(\mathbf{U}+\mathbf{v})\cdot\nabla\mathbf{U} - \mu_1\Delta\mathbf{U} - \mu_2\nabla\operatorname{div}\mathbf{U} + \nabla P &= \bar{\rho}\mathbf{f},\\
\mathbf{U}|_{\partial\Omega} &= \mathbf{0},\\
\frac{1}{|\Omega|}\int_\Omega P dx &= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{div}(\bar{\rho}\mathbf{v}) + \epsilon^2\operatorname{div}(\sigma(\mathbf{U}+\mathbf{v})) &= \epsilon^2 G(\sigma,\mathbf{v},\mathbf{U},P),\\
\bar{\rho}(\mathbf{U}+\mathbf{v})\cdot\nabla\mathbf{v} - \mu_1\Delta\mathbf{v} - \lambda_1\nabla\operatorname{div}\mathbf{v} + \nabla\sigma &= \epsilon^2\mathbf{F}(\sigma,\mathbf{v},\mathbf{U},P),\\
\mathbf{u}|_{\partial\Omega} &= \mathbf{0},\\
\frac{1}{|\Omega|}\int_\Omega \sigma dx &= 0.
\end{aligned}
$$

Linearlization

$$\begin{aligned} \mathrm{div}(\bar\rho\mathbf{U}) &= 0, \\ \bar\rho(\tilde{\mathbf{U}}+\tilde{\mathbf{v}})\cdot\nabla\mathbf{U} - \mu_1\Delta\mathbf{U} - \mu_2\nabla\mathrm{div}\mathbf{U} + \nabla P = \bar\rho\mathbf{f}&, \\ \mathbf{U}|_{\partial\Omega} &= \mathbf{0}, \\ \frac{1}{|\Omega|}\int_\Omega P dx &= 0, \end{aligned}$$

and

$$\begin{aligned} (\sigma-\tilde\sigma) + \mathrm{div}(\bar\rho\mathbf{v}) + \epsilon^2\mathrm{div}(\sigma(\mathbf{U}+\mathbf{v})) &= \epsilon^2 G(\tilde\sigma,\tilde{\mathbf{v}},\tilde{\mathbf{U}},\tilde P), \\ \bar\rho(\tilde{\mathbf{U}}+\tilde{\mathbf{v}})\cdot\nabla\mathbf{v} - \mu_1\Delta\mathbf{v} - \lambda_1\nabla\mathrm{div}\mathbf{v} + \nabla\sigma &= \epsilon^2\mathbf{F}(\tilde\sigma,\tilde{\mathbf{v}},\tilde{\mathbf{U}},\tilde P), \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{0}, \\ \frac{1}{|\Omega|}\int_\Omega \sigma dx &= 0. \end{aligned}$$

Introduce an operator $T : X \to X$, $(\tilde{\mathbf{U}},\tilde P,\tilde\sigma,\tilde{\mathbf{v}}) \mapsto (\mathbf{U},P,\sigma,\mathbf{v})$. Let $X = \{(\mathbf{U},\mathbf{v},P,\sigma) \in H_0^3\times H_0^3\times H^2\times H^2 : \|P$
${}_{H^2} + \|\sigma\|_{H^2} + \|\mathbf{U}\|_{H^3} + \|\mathbf{v}\|_{H^3} \le M\}$. To apply Schauder theorem, we should find $M>0$ so that if $(\tilde P,\tilde\sigma,\tilde{\mathbf{U}},\tilde{\mathbf{u}}) \in X$, then $T(\tilde P,\tilde\sigma,\tilde{\mathbf{U}},\tilde{\mathbf{u}}) = (\tilde P,\sigma,\tilde{\mathbf{U}},\mathbf{v}) \in X$.

By standard estimates we can show that there is small $M>0$ and ϵ_0 so that $TX\subset X$ if $\epsilon\le\epsilon_0$. We can also show that T is continuous with respect to the norm of the Banach space $L^2\times L^2\times H_0^1\times H_0^1$. Finally, applying Schauder's compactness theorem there is a fixed point of T.

Defect: Uniqueness? Incompressible limit?

Try to find better decomposition or better approximation!!!

Weak solution for large data to the equations

$$\begin{aligned} \mathrm{div}(\rho\mathbf{u}) &= 0, \\ \rho\mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} - \lambda\nabla\mathrm{div}\mathbf{u} + \nabla p &= \rho\mathbf{f}, \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{0}. \end{aligned}$$

By P.L. Lions(1997) for large data when $\gamma > \max\{3, \frac{n}{2}\}$,

Defect: Uniqueness, Regularity?

Search for the recent literature concerning the cases $\gamma \leq \frac{n}{2}$.

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