An Introduction to Mathematical Modelling in Fluid Mechanics and Theory of the Navier-Stokes Equations

Jiří Neustupa

Mathematical Institute of the Czech Academy of Sciences

Lecture 3:

Non-steady Navier-Stokes equations I

(weak solutions, global in time existence)

Yonsei University, Seoul, December 2019

Contents

1. Notation, operator $\mathcal{P}_\sigma$

Notation. • The outer normal vector to $\partial\Omega$ is denoted by $\mathbf{n}$.

- Vector functions and spaces of vector functions are denoted by boldface letters.
- $\mathbf{C}^{\infty}_{0,\sigma}(\Omega)$ denotes the linear space of infinitely differentiable divergence-free vector functions in Ω, with a compact support in Ω.
- $\mathbf{L}^2_\sigma(\Omega)$ is the closure of $\mathbf{C}^{\infty}_{0,\sigma}(\Omega)$ in $\mathbf{L}^2(\Omega)$.
- $\mathbf{W}^{1,2}_{0,\sigma}(\Omega) := \mathbf{L}^2_\sigma(\Omega) \cap \mathbf{W}^{1,2}_0(\Omega)$
- The norm in $L^q(\Omega)$ and in $\mathbf{L}^q(\Omega)$ is denoted by $\| \, . \, \|_q$. The norm in $W^{k,q}(\Omega)$ and in $\mathbf{W}^{k,q}(\Omega)$ (for $k \in \mathbb{N}$) is denoted by $\| \, . \, \|_{k,q}$.

  If the considered domain differs from Ω then we use e.g. the notation $\| \, . \, \|_{q;\,\Omega'}$, etc.

  The scalar product in $L^2(\Omega)$ and in $\mathbf{L}^2(\Omega)$ is denoted by $(\, . \, , \, . \,)_2$ and the scalar product in $W^{1,2}(\Omega)$ and in $\mathbf{W}^{1,2}(\Omega)$ is denoted by $(\, . \, , \, . \,)_{1,2}$.
- The duality between elements of $\mathbf{W}^{-1,2}_0(\Omega)$ and $\mathbf{W}^{1,2}_0(\Omega)$ is denoted by $\langle \, . \, , \, . \, \rangle_0$ and the duality between elements of $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ and $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ is denoted by $\langle \, . \, , \, . \, \rangle_{0,\sigma}$.

$\mathbf{L}^2(\Omega)$ and $\mathbf{L}^2_\sigma(\Omega)$ as subspaces of $\mathbf{W}_0^{-1,2}(\Omega)$ and $\mathbf{W}_{0,\sigma}^{-1,2}(\Omega)$. The Lebesgue space $\mathbf{L}^2(\Omega)$ can be identified with a subspace of $\mathbf{W}_0^{-1,2}(\Omega)$ so that if $\mathbf{f} \in \mathbf{L}^2(\Omega)$ then

$$\langle \mathbf{f}, \mathbf{w} \rangle_0 \ := \ \int_\Omega \mathbf{f} \cdot \mathbf{w} \, \mathrm{d}\mathbf{x} \qquad \text{for all } \mathbf{w} \in \mathbf{W}_0^{1,2}(\Omega). \tag{1}$$

Similarly, $\mathbf{L}^2_\sigma(\Omega)$ can be identified with a subspace of $\mathbf{W}_{0,\sigma}^{-1,2}(\Omega)$ so that if $\mathbf{f} \in \mathbf{L}^2_\sigma(\Omega)$ then

$$\langle \mathbf{f}, \mathbf{w} \rangle_{0,\sigma} \ := \ \int_\Omega \mathbf{f} \cdot \mathbf{w} \, \mathrm{d}\mathbf{x} \qquad \text{for all } \mathbf{w} \in \mathbf{W}_{0,\sigma}^{1,2}(\Omega). \tag{2}$$

Note that $\mathbf{L}^2(\Omega)$ cannot be identified with a subspace of $\mathbf{W}_{0,\sigma}^{-1,2}(\Omega)$.

Operator $\mathcal{P}_\sigma$. $\mathbf{W}_{0,\sigma}^{1,2}(\Omega)$ is a closed subspace of $\mathbf{W}_0^{1,2}(\Omega)$. If $\mathbf{f} \in \mathbf{W}_0^{-1,2}(\Omega)$ (i.e. $\mathbf{f}$ is a bounded linear functional on $\mathbf{W}_0^{1,2}(\Omega)$) then we denote by $\mathcal{P}_\sigma \mathbf{f}$ the restriction of $\mathbf{f}$ to $\mathbf{W}_{0,\sigma}^{1,2}(\Omega)$. Thus, $\mathcal{P}_\sigma \mathbf{f}$ is an element of $\mathbf{W}_{0,\sigma}^{-1,2}(\Omega)$, defined by the equation

$$\langle \mathcal{P}_\sigma \mathbf{f}, \mathbf{w} \rangle_{0,\sigma} \ := \ \langle \mathbf{f}, \mathbf{w} \rangle_0 \qquad \text{for all } \mathbf{w} \in \mathbf{W}_{0,\sigma}^{1,2}(\Omega).$$

Obviously, $\mathcal{P}_\sigma$ is a linear operator from $\mathbf{W}_0^{-1,2}(\Omega)$ to $\mathbf{W}_{0,\sigma}^{-1,2}(\Omega)$, whose domain is the whole space $\mathbf{W}_0^{-1,2}(\Omega)$.

Lemma 1. *Operator $\mathcal{P}_\sigma$ is bounded, its range is $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ and $\mathcal{P}_\sigma$ is not 1–1.*

Proof. 1) The proof of the boundedness of $\mathcal{P}_\sigma$ is a simple exercise: let $\mathbf{f} \in \mathbf{W}^{-1,2}_0(\Omega)$. Then

$$\|\mathcal{P}_\sigma \mathbf{f}\|_{-1,2;\,\sigma} = \sup_{\mathbf{w}\in\mathbf{W}^{1,2}_{0,\sigma}(\Omega);\ \mathbf{w}\neq\mathbf{0}} \frac{|\langle \mathcal{P}_\sigma \mathbf{f}, \mathbf{w}\rangle_{0,\sigma}|}{\|\mathbf{w}\|_{1,2}} = \sup_{\mathbf{w}\in\mathbf{W}^{1,2}_{0,\sigma}(\Omega);\ \mathbf{w}\neq\mathbf{0}} \frac{|\langle \mathbf{f}, \mathbf{w}\rangle_0|}{\|\mathbf{w}\|_{1,2}}$$
$$\leq \sup_{\mathbf{w}\in\mathbf{W}^{1,2}_{0}(\Omega);\ \mathbf{w}\neq\mathbf{0}} \frac{|\langle \mathbf{f}, \mathbf{w}\rangle_0|}{\|\mathbf{w}\|_{1,2}} = \|\mathbf{f}\|_{-1,2}.$$

2) Let $\mathbf{g} \in \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. There exists (by the Hahn-Banach theorem) an extension of $\mathbf{g}$ from $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{1,2}_0(\Omega)$, which we denote by $\mathbf{g}_{\mathrm{ext}}$. The extension is an element of $\mathbf{W}^{-1,2}_0(\Omega)$, satisfying $\|\mathbf{g}_{\mathrm{ext}}\|_{-1,2} = \|\mathbf{g}\|_{-1,2;\,\sigma}$ and

$$\langle \mathbf{g}_{\mathrm{ext}}, \mathbf{w}\rangle_0 = \langle \mathbf{g}, \mathbf{w}\rangle_{0,\sigma} \qquad \text{for all } \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega).$$

This shows that $\mathbf{g} = \mathcal{P}_\sigma(\mathbf{g}_{\mathrm{ext}})$. Consequently, $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega) = R(\mathcal{P}_\sigma)$.

3) Finally, taking $\mathbf{f} = \nabla g$ for $g \in C_0^\infty(\Omega)$, we get

$$\langle \mathcal{P}_\sigma \mathbf{f}, \mathbf{w} \rangle_{0,\sigma} \;=\; \langle \mathbf{f}, \mathbf{w} \rangle_0 \;=\; \int_\Omega \nabla q \cdot \mathbf{w} \, \mathrm{d}\mathbf{x} \;=\; 0$$

for all $\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. This shows that the operator $\mathcal{P}_\sigma$ is not one-to-one. ■

The relation between the Helmholtz projection P_σ and operator $\mathcal{P}_\sigma$. Let $\mathbf{g} \in \mathbf{L}^2(\Omega)$. Treating $\mathbf{g}$ as an element of $\mathbf{W}_0^{-1,2}(\Omega)$, we have

$$\langle \mathcal{P}_\sigma \mathbf{g}, \mathbf{w} \rangle_{0,\sigma} \;=\; \langle \mathbf{g}, \mathbf{w} \rangle_0 \qquad \text{for all } \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega).$$

Writing $\mathbf{g} = P_\sigma \mathbf{g} + Q_\sigma \mathbf{g}$, we also get

$$\langle \mathbf{g}, \mathbf{w} \rangle_0 \;=\; \big\langle P_\sigma \mathbf{g} + Q_\sigma \mathbf{g}, \mathbf{w} \big\rangle_0 \;=\; \big\langle P_\sigma \mathbf{g}, \mathbf{w} \big\rangle_0 \;=\; \big\langle P_\sigma \mathbf{g}, \mathbf{w} \big\rangle_{0,\sigma} \qquad \text{for all } \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega).$$

(The last equality follows from formulas (1) and (2).) Consequently,

$$\langle \mathcal{P}_\sigma \mathbf{g}, \mathbf{w} \rangle_{0,\sigma} \;=\; \langle P_\sigma \mathbf{g}, \mathbf{w} \rangle_{0,\sigma} \qquad \text{for all } \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega).$$

Hence $\mathcal{P}_\sigma \mathbf{g}$ and $P_\sigma \mathbf{g}$ represent the same element of $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. As $P_\sigma \mathbf{g} \in \mathbf{L}^2_\sigma(\Omega)$, $\mathcal{P}_\sigma \mathbf{g}$ can also be considered to be an element of $\mathbf{L}^2_\sigma(\Omega)$. In this sense, we observe that *the Helmholtz projection P_σ coincides with the restriction of $\mathcal{P}_\sigma$ to $\mathbf{L}^2(\Omega)$.*

An explicit expression of $\mathcal{P}_\sigma$.

It is proven in [8] that $\mathcal{P}_\sigma = \mathcal{S}_2^{-1}\,\mathcal{Q}_2$, where

$\mathcal{S}_2 : \mathbf{W}_{0,\sigma}^{-1,2}(\Omega) \to \mathbf{W}_0^{-1,2}(\Omega)|_{\mathbf{W}_{0,\sigma}^{1,2}(\Omega)^\perp}$ is a mapping, defined by the formula

$$\mathcal{S}_2(\mathbf{g}) \;:=\; \mathbf{g}_{\mathrm{ext}} + \mathbf{W}_{0,\sigma}^{1,2}(\Omega)^\perp \qquad \text{for } \mathbf{g} \in \mathbf{W}_{0,\sigma}^{-1,2}(\Omega).$$

$\mathbf{g}_{\mathrm{ext}}$ ... an extension of the functional $\mathbf{g}$ (acting on $\mathbf{W}_{0,\sigma}^{1,2}(\Omega)$ to a functional (acting on $\mathbf{W}_0^{1,2}(\Omega)$. (The extension exists due to the Hahn–Banach theorem.) One can show that $\mathcal{S}_2(\mathbf{g})$ is independent of a concrete choice of the extension $\mathbf{g}_{\mathrm{ext}}$.

$\mathcal{S}_2$ is an isometric isomorphism of $\mathbf{W}_{0,\sigma}^{-1,2}(\Omega)$ onto $\mathbf{W}_0^{-1,2}(\Omega)|_{\mathbf{W}_{0,\sigma}^{1,2}(\Omega)^\perp}$, see [8].

$\mathcal{Q}_2 : \mathbf{W}_0^{-1,2}(\Omega) \to \mathbf{W}_0^{-1,2}(\Omega)|_{\mathbf{W}_{0,\sigma}^{1,2}(\Omega)^\perp}$ is the so called *quotient mapping*, defined by the formula

$$\mathcal{Q}_2(\mathbf{f}) \;:=\; \mathbf{f} + \mathbf{W}_{0,\sigma}^{1,2}(\Omega)^\perp \qquad \text{for } \mathbf{f} \in \mathbf{W}_0^{-1,2}(\Omega).$$

2. The Navier–Stokes initial–boundary value problem and its weak formulation

Recall that Ω is a domain in $\mathbb{R}^3$, $T > 0$. We denote $Q_T := \Omega \times (0,T)$ and $\Gamma_T := \partial\Omega \times (0,T)$.

A classical form of the Navier-Stokes IBVP. The Navier-Stokes IBVP (i.e. initial–boundary value problem) is given by the equations

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \nu\Delta\mathbf{u} + \mathbf{f} \qquad \text{in } Q_T, \tag{3}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } Q_T, \tag{4}$$

the boundary condition

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_T \tag{5}$$

and the initial condition

$$\mathbf{u} = \mathbf{u}_0 \qquad \text{in } \Omega \times \{0\}. \tag{6}$$

The unknowns are $\mathbf{u}$ (velocity) and p (pressure). Function $\mathbf{f}$ represents an external body force and ν is the kinematic coefficient of viscosity. It is supposed to be a positive constant.

A priori estimates. Assume that $\mathbf{u}$ is a solution of the problem (3)–(6), as smooth as we need. Multiply equation (6) by $\mathbf{u}$ and integrate in Ω. We get:

$$\int_\Omega \partial_t \mathbf{u}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} \;=\; \int_\Omega \partial_t\,\frac{1}{2}\,(\mathbf{u}\cdot\mathbf{u})\,\mathrm{d}\mathbf{x} \;=\; \frac{1}{2}\int_\Omega \partial_t\,|\mathbf{u}|^2\,\mathrm{d}\mathbf{x} \;=\; \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\mathbf{u}|^2\,\mathrm{d}\mathbf{x},$$

$$\begin{aligned}\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} \;&=\; \int_\Omega u_j\,(\partial_j u_i)\,u_i\,\mathrm{d}\mathbf{x} \;=\; \int_\Omega u_j\,\partial_j\left(\frac{1}{2}\,u_iu_i\right)\mathrm{d}\mathbf{x}\\ &=\; -\int_\Omega (\partial_j u_j)\,\frac{1}{2}\,u_iu_i\,\mathrm{d}\mathbf{x} \;=\; 0,\end{aligned}$$

$$\int_\Omega \nabla p\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} \;=\; \int_\Omega (\partial_i p)\,u_i\,\mathrm{d}\mathbf{x} \;=\; -\int_\Omega p\,(\partial_i u_i)\,\mathrm{d}\mathbf{x} \;=\; -\int_\Omega p\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}\mathbf{x} \;=\; 0,$$

$$\int_\Omega \Delta\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} \;=\; \int_\Omega (\partial_k\partial_k u_i)\,u_i\,\mathrm{d}\mathbf{x} \;=\; -\int_\Omega (\partial_k u_i)\,(\partial_k u_i)\,\mathrm{d}\mathbf{x} \;=\; -\int_\Omega |\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x}.$$

Hence we obtain

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\mathbf{u}|^2\,\mathrm{d}\mathbf{x} + \nu\int_\Omega |\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x} \;=\; \int_\Omega \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}. \tag{7}$$

Assume, for simplicity, that domain Ω is bounded. In this case, the norms $\|\,.\,\|_{1,2}$ and $\|\nabla.\,\|_2$ are equivalent in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$.

Treating the right hand side as the duality $\langle \mathbf{f}, \mathbf{u}\rangle_0$, we can estimate it:

$$\big\langle \mathbf{f}, \mathbf{u}\big\rangle_0 \;\leq\; \|\mathbf{f}\|_{-1,2}\,\|\mathbf{u}\|_{1,2} \;\leq\; c_1\,\|\mathbf{f}\|_{-1,0}\,\|\nabla\mathbf{u}\|_2 \;\leq\; \frac{\nu}{2}\,\|\nabla\mathbf{u}\|_2^2 + \frac{c_1^2}{2\nu}\,\|\mathbf{f}\|_{-1,2}^2.$$

Substituting this to (7), we get

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|\mathbf{u}(t)\|_2^2 + \frac{\nu}{2}\,\|\nabla\mathbf{u}(t)\|_2^2 \;\leq\; \frac{c_1^2}{2\nu}\,\|\mathbf{f}\|_{-1,2}^2,$$

$$\|\mathbf{u}(t)\|_2^2 + \nu\int_0^t \|\nabla\mathbf{u}(\tau)\|_2^2\,\mathrm{d}\tau \;\leq\; \frac{c_1^2}{\nu}\int_0^t \|\mathbf{f}(\tau)\|_{-1,2}^2\,\mathrm{d}\tau \;\leq\; \frac{c_1^2}{\nu}\int_0^T \|\mathbf{f}(\tau)\|_{-1,2}^2\,\mathrm{d}\tau \;=:\; M.$$

This implies:

$$\|\mathbf{u}(t)\|_2 \;\leq\; \sqrt{M} \quad \text{for all } t\in(0,T), \tag{8}$$

$$\int_0^T \|\nabla\mathbf{u}(\tau)\|_2^2\,\mathrm{d}\tau \;\leq\; M. \tag{9}$$

These inequalities indicate which are the reasonable spaces for a weak solution:
$\mathbf{u}\in L^\infty(0,T;\,\mathbf{L}^2_\sigma(\Omega))$ (from (8)) and $\mathbf{u}\in L^2(0,T;\,\mathbf{W}^{1,2}_{0,\sigma}(\Omega))$ (from (9)).

The 1st weak formulation of the Navier-Stokes IBVP (3)–(6).

Given $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\Omega)$ *and* $\mathbf{f} \in L^2(0,T;\, \mathbf{W}_0^{-1,2}(\Omega))$*. A function* $\mathbf{u} \in L^\infty(0,T;\; \mathbf{L}^2_\sigma(\Omega)) \cap L^2(0,T;\; \mathbf{W}^{1,2}_{0,\sigma}(\Omega))$ *is said to be a weak solution to the problem (3)–(6) if*

$$\int_0^T \int_\Omega \big[-\mathbf{u}\cdot\partial_t\boldsymbol{\phi} + \nu\nabla\mathbf{u} : \nabla\boldsymbol{\phi} + \mathbf{u}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\phi}\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_\Omega \mathbf{u}_0\cdot\boldsymbol{\phi}(\mathbf{x},0)\,\mathrm{d}\mathbf{x} + \int_0^T \langle\mathbf{f},\boldsymbol{\phi}\rangle_0\,\mathrm{d}t \tag{10}$$

for all $\boldsymbol{\phi} \in C^\infty\big([0,T];\; \mathbf{W}^{1,2}_{0,\sigma}(\Omega)\big)$ *such that* $\boldsymbol{\phi}(T) = \mathbf{0}$.

Equation (10) follows from (3)–(6) if one formally multiplies equation (3) by the test function $\boldsymbol{\phi}$, applies the integration by parts and uses the boundary condition (5) and the initial condition (6).

As the integral of $\nabla p\cdot\boldsymbol{\phi}$ vanishes, the pressure p does not explicitly appear in (10). However, one can show that a certain pressure (at least as a distribution) can always be assigned to every weak solution. (See the section on the associated pressure.)

The 2nd weak formulation of the Navier–Stokes IBVP (3)–(6).

We define the operators $\mathcal{A}: \mathbf{W}_0^{1,2}(\Omega) \to \mathbf{W}_0^{-1,2}(\Omega)$ and $\mathcal{B}: \left[\mathbf{W}_0^{1,2}(\Omega)\right]^2 \to \mathbf{W}_0^{-1,2}(\Omega)$ by the equations

$$\begin{aligned}
\big\langle \mathcal{A}\mathbf{v}, \mathbf{z} \big\rangle_0 \; &:= \int_\Omega \nabla\mathbf{v} : \nabla\mathbf{z} \, \mathrm{d}\mathbf{x} && \text{for } \mathbf{v}, \mathbf{z} \in \mathbf{W}_0^{1,2}(\Omega), \\
\big\langle \mathcal{B}(\mathbf{v}, \mathbf{w}), \mathbf{z} \big\rangle_0 \; &:= \int_\Omega \mathbf{v} \cdot \nabla\mathbf{w} \cdot \mathbf{z} \, \mathrm{d}\mathbf{x} && \text{for } \mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathbf{W}_0^{1,2}(\Omega).
\end{aligned}$$

Operator $\mathcal{A}$ is bounded and 1–1. It is related to operator $\mathcal{A}_\sigma$, introduced in Lecture 2, through the formula $\mathcal{A}_\sigma = \mathcal{P}_\sigma \mathcal{A}$.

The bilinear operator $\mathcal{B}$ satisfies

$$\begin{aligned}
\|\mathcal{B}(\mathbf{v}, \mathbf{w})\|_{-1,2} \; &= \sup_{\mathbf{z}\in\mathbf{W}_0^{1,2}(\Omega),\; \mathbf{z}\neq\mathbf{0}} \frac{|\langle \mathcal{B}(\mathbf{v}, \mathbf{w}), \mathbf{z} \rangle_\Omega|}{\|\mathbf{z}\|_{1,2}} \\
&= \sup_{\mathbf{z}\in\mathbf{W}_0^{1,2}(\Omega),\; \mathbf{z}\neq\mathbf{0}} \frac{|(\mathbf{v}\cdot\nabla\mathbf{w},\, \mathbf{z})_2|}{\|\mathbf{z}\|_{1,2}} \leq \sup_{\mathbf{z}\in\mathbf{W}_0^{1,2}(\Omega),\; \mathbf{z}\neq\mathbf{0}} \frac{\|\mathbf{v}\|_2^{1/2} \, \|\mathbf{v}\|_6^{1/2} \, \|\nabla\mathbf{w}\|_2 \, \|\mathbf{z}\|_6}{\|\mathbf{z}\|_{1,2}} \\
&\leq \; c \, \|\mathbf{v}\|_2^{1/2} \, \|\nabla\mathbf{v}\|_2^{1/2} \, \|\nabla\mathbf{w}\|_2. \qquad (11)
\end{aligned}$$

Let $\mathbf{u}$ be a weak solution of the IBVP (3)–(6) in the sense of the 1st definition. It follows from the boundedness of $\mathcal{A}$ from $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ and estimates (11) that

$$\mathcal{A}\mathbf{u} \in L^2(0,T;\, \mathbf{W}_0^{-1,2}(\Omega)) \quad \text{and} \quad \mathcal{B}(\mathbf{u},\mathbf{u}) \in L^{4/3}(0,T;\, \mathbf{W}_0^{-1,2}(\Omega)). \tag{12}$$

Considering $\boldsymbol{\phi}$ in (10) in the form $\boldsymbol{\phi}(\mathbf{x},t) = \mathbf{w}(\mathbf{x})\,\vartheta(t)$ where $\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ and $\vartheta \in C_0^\infty((0,T))$, we deduce that $\mathbf{u}$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{u},\mathbf{w})_2 + \nu\,\big\langle \mathcal{A}\mathbf{u},\mathbf{w}\big\rangle_0 + \big\langle \mathcal{B}(\mathbf{u},\mathbf{u}),\mathbf{w}\big\rangle_0 \;=\; \langle \mathbf{f},\mathbf{w}\rangle_0 \qquad \text{a.e. in } (0,T), \tag{13}$$

where the derivative of $(\mathbf{u},\boldsymbol{\varphi})_2$ means the derivative in the sense of distributions.

It follows from (12) that $\langle \mathcal{A}\mathbf{u},\mathbf{w}\rangle_0 \in L^2(0,T)$ and $\langle \mathcal{B}(\mathbf{u},\mathbf{u}),\mathbf{w}\rangle_0 \in L^{4/3}(0,T)$. Since $\langle \mathbf{f},\mathbf{w}\rangle_0 \in L^2(0,T)$, we obtain from (13) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{u},\mathbf{w})_2 \quad \text{(the distributional derivative)} \quad \in L^{4/3}(0,T).$$

Hence $(\mathbf{u},\mathbf{w})_2$ is a.e. in $[0,T)$ equal to a continuous function and the weak solution $\mathbf{u}$ is (after a possible redefinition on a set of measure zero) a weakly continuous function from $[0,T)$ to $\mathbf{L}^2_\sigma(\Omega)$.

Now, we deduce that $\mathbf{u}$ satisfies the initial condition (6) in this sense:

$$(\mathbf{u}, \mathbf{w})_2\Big|_{t=0} = (\mathbf{u}_0, \mathbf{w})_2 \tag{14}$$

for all $\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$.

We come to the 2nd weak formulation of the IBVP (3)–(6):

Given $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\Omega)$ *and* $\mathbf{f} \in L^2(0,T;\ \mathbf{W}_0^{-1,2}(\Omega))$. *Find* $\mathbf{u} \in L^\infty(0,T;\ \mathbf{L}^2_\sigma(\Omega)) \cap L^2(0,T;\ \mathbf{W}^{1,2}_{0,\sigma}(\Omega))$, *called the weak solution, such that* $\mathbf{u}$ *satisfies the equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}, \mathbf{w})_2 + \nu \left\langle \mathcal{A}\mathbf{u}, \mathbf{w} \right\rangle_0 + \left\langle \mathcal{B}(\mathbf{u}, \mathbf{u}), \mathbf{w} \right\rangle_0 = \langle \mathbf{f}, \mathbf{w} \rangle_0 \qquad \text{a.e. in } (0,T) \tag{10}$$

and the initial condition

$$(\mathbf{u}, \mathbf{w})_2\Big|_{t=0} = (\mathbf{u}_0, \mathbf{w})_2 \tag{11}$$

for all $\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$.

We have shown that if $\mathbf{u}$ is a weak solution of the IBVP (3)–(6) in the sense of the 1st definition then it also satisfies the 2nd definition.

One can also show the opposite, i.e. if $\mathbf{u}$ satisfies the 2nd definition then it also satisfies the 1st definition.

For that purpose, it is sufficient to take into account that any test function $\boldsymbol{\phi}$ in (10) can be approximated with an arbitrarily small error (measured in the norm of $C^1\big([0,T];\mathbf{W}^{1,2}_{0,\sigma}(\Omega)\big)$) by a finite linear combination of functions of the type

$$\mathbf{w}(\mathbf{x})\,\vartheta(t),$$

where $\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ and $\vartheta \in C^\infty([0,T])$, $\vartheta(T) = 0$, and that each such pair $\mathbf{w}$, ϑ satisfies the equation

$$\begin{aligned}\int_0^T \big[-(\mathbf{u},\mathbf{w})_2\,\vartheta'(t) + \nu\,\langle\mathcal{A}\mathbf{u},\mathbf{w}\rangle_0\,\vartheta(t) + \langle\mathcal{B}(\mathbf{u},\mathbf{u}),\mathbf{w}\rangle_0\,\vartheta(t)\big]\;\mathrm{d}t \\ = \;(\mathbf{u}_0,\mathbf{w})_2\,\vartheta(0) + \int_0^T \langle\mathbf{f},\mathbf{w}\rangle_0\,\vartheta(t)\;\mathrm{d}t,\end{aligned}$$

which follows from (13).

An important lemma. Before we proceed with another weak formulation of the IBVP (3)–(6), we present a lemma, which coincides with Lemma III.1.1 in [10]:

Lemma 2. *Let* $\mathbf{X}$ *be a Banach space with the dual* $\mathbf{X}^*$*,* $\langle\, .\,,\, .\,\rangle$ *be the duality between* $\mathbf{X}^*$ *and* $\mathbf{X}$*,* $-\infty < a < b < \infty$ *and* $\mathbf{u}$*,* $\mathbf{g} \in L^1(a,b;\, \mathbf{X})$*. Then the following three conditions are equivalent:*

1) $\mathbf{u}$ *is a.e. in* (a,b) *equal to a primitive function of* $\mathbf{g}$*, which means that*

$$\mathbf{u}(t) = \boldsymbol{\xi} + \int_a^t \mathbf{g}(s)\,\mathrm{d}s \quad \textit{for some } \boldsymbol{\xi} \in \mathbf{X} \textit{ and a.a. } t \in (a,b),$$

2) $\displaystyle\int_a^b \vartheta'(t)\,\mathbf{u}(t)\,\mathrm{d}t = -\int_a^b \vartheta(t)\,\mathbf{g}(t)\,\mathrm{d}t \quad \textit{for all } \vartheta \in C_0^\infty((a,b)),$

3) $\displaystyle\frac{\mathrm{d}}{\mathrm{d}t}\,\langle \boldsymbol{\eta}, \mathbf{u}\rangle = \langle \boldsymbol{\eta}, \mathbf{g}\rangle \quad \textit{in the sense of distributions in } (a,b) \textit{ for each } \boldsymbol{\eta} \in \mathbf{X}^*.$

If the conditions 1) – 3) are fulfilled then $\mathbf{u}$ *is a.e. in* (a,b) *equal to a continuous function from* $[a,b]$ *to* $\mathbf{X}$*.*

Note that if functions $\mathbf{u}$ and $\mathbf{g}$ are related as in item 2) then $\mathbf{g}$ is called the *distributional derivative* of $\mathbf{u}$ with respect to t and it is usually denoted by $\mathbf{u}'$.

The 3rd weak formulation of the Navier-Stokes IBVP (3)–(6).

Equation (13) can also be written in the equivalent form

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u},\mathbf{w})_2 + \nu\left\langle \mathcal{A}_\sigma\mathbf{u},\mathbf{w}\right\rangle_{0,\sigma} + \left\langle \mathcal{P}_\sigma\mathcal{B}(\mathbf{u},\mathbf{u}),\mathbf{w}\right\rangle_{0,\sigma} \;=\; \left\langle \mathcal{P}_\sigma\mathbf{f},\mathbf{w}\right\rangle_{0,\sigma}. \tag{15}$$

Let us denote by $(\mathbf{u}')_\sigma$ the distributional derivative with respect to t of $\mathbf{u}$, as a function from $(0,T)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$.

Applying Lemma 2 (with $\mathbf{X}=\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ and $\mathbf{X}^*=\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$), we deduce that equation (15) is equivalent to

$$(\mathbf{u}')_\sigma + \nu\mathcal{A}_\sigma\mathbf{u} + \mathcal{P}_\sigma\mathcal{B}(\mathbf{u},\mathbf{u}) \;=\; \mathcal{P}_\sigma\mathbf{f}, \tag{16}$$

which is an equation in $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$, satisfied a.e. in the time interval $(0,T)$. Due to (12), $(\mathbf{u}')_\sigma\in L^{4/3}(0,T;\,\mathbf{W}^{-1,2}_{0,\sigma}(\Omega))$. Hence $\mathbf{u}$ coincides a.e. in $(0,T)$ with a continuous function from $[0,T)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. We obtain the 3rd equivalent definition of a weak solution to the IBVP (3)–(6):

Given $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\Omega)$ *and* $\mathbf{f} \in L^2(0,T;\ \mathbf{W}_0^{-1,2}(\Omega))$. *Function* $\mathbf{u} \in L^\infty(0,T;\ \mathbf{L}^2_\sigma(\Omega)) \cap L^2(0,T;\ \mathbf{W}^{1,2}_{0,\sigma}(\Omega))$ *is called a weak solution to the IBVP (3)–(6) if* $\mathbf{u}$ *satisfies the equation*

$$(\mathbf{u}')_\sigma + \nu \mathcal{A}_\sigma \mathbf{u} + \mathcal{P}_\sigma \mathcal{B}(\mathbf{u},\mathbf{u}) \ = \ \mathcal{P}_\sigma \mathbf{f}, \tag{13}$$

a.e. in the interval $(0,T)$ *and the initial condition*

$$\mathbf{u}\,\Big|_{t=0} \ = \ \mathbf{u}_0, \tag{6}$$

where $\mathbf{u}|_{t=0}$ *is the value of the aforementioned continuous function at time* $t=0$.

We have explained that if $\mathbf{u}$ is a weak solution in the sense of the 2nd definition then it satisfies the 3rd definition. The validity of the opposite implication can be verified by means of Lemma 2.

Remark. We have shown that $\mathbf{u}$ coincides a.e. in $(0,T)$ with a continuous function from $[0,T)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. This, however, does not imply that $\mathbf{u}$ coincides a.e. in $(0,T)$ with a continuous function from $[0,T)$ to $\mathbf{W}_0^{-1,2}(\Omega)$.

(It is because $(\mathbf{u}')_\sigma$ is the distributional derivative with respect to t of $\mathbf{u}$, as a function from $(0,T)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$, and not the distributional derivative with respect to t of $\mathbf{u}$, as a function from $(0,T)$ to $\mathbf{W}^{-1,2}_{0}(\Omega)$.)

As it is important to distinguish between these two derivatives, we use the different notation: while the first derivative is denoted by $(\mathbf{u}')_\sigma$, the second is denoted just by $\mathbf{u}'$. We can formally write: $(\mathbf{u}')_\sigma = \mathcal{P}_\sigma \mathbf{u}'$.

3. Existence of a Leray–Hopf weak solution to the Navier–Stokes IBVP (3)–(6)

Theorem 1 (Leray 1934, Hopf 1951, et al). *There exists at least one weak solution* $\mathbf{u}$ *of the Navier–Stokes IBVP (3)–(6). The solution satisfies the energy inequality*

$$\|\mathbf{u}(\,.\,,t)\|_2^2 + 2\nu \int_0^t \|\nabla\mathbf{u}(\,.\,,\tau)\|_2^2 \, \mathrm{d}\tau \;\leq\; \|\mathbf{u}_0\|_2^2 + 2\int_0^t \big\langle \mathbf{f}(\tau), \mathbf{u}(\,.\,\tau)\big\rangle_0 \, \mathrm{d}\tau \qquad (17)$$

for all $t \in [0,T)$, *and*

$$\lim_{t\to 0+} \|\mathbf{u}(\,.\,,t) - \mathbf{u}_0\|_2 \;=\; 0. \qquad (18)$$

4. Principles of the proof of Theorem 1

Recall that A_σ is a self-adjoint positive operator in $\mathbf{L}^2_\sigma(\Omega)$. Assume in this section, for simplicity, that domain Ω is bounded and Lipschitzian. Then $\mathbf{W}^{1,2}_{0,\sigma}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^2_\sigma(\Omega)$. Consequently, A_σ is an operator with compact resolvent.

In this case, the spectrum of A_σ consists of an increasing sequence of infinitely many isolated positive eigenvalues, each of whose has a finite multiplicity. (See Lemma 10 in Lecture 2.) The eigenvalues can be ordered to a sequence

$$\lambda_1 \ \leq \ \lambda_2 \ \leq \ \lambda_3 \ \leq \lambda_4 \ \leq \ \ldots$$

so that each eigenvalue λ appears in the sequence as many times, as is the multiplicity of λ. The corresponding eigenfunctions

$$\boldsymbol{\varphi}_1, \ \boldsymbol{\varphi}_2, \ \boldsymbol{\varphi}_3, \ \ldots$$

can be chosen so that they form a complete ortho-normal system in $\mathbf{L}^2_\sigma(\Omega)$.

For $n \in \mathbb{N}$, put $\boldsymbol{\mathcal{V}}_n := \mathcal{L}\{\boldsymbol{\varphi}_1, \ \boldsymbol{\varphi}_2, \ \ldots, \ \boldsymbol{\varphi}_n\}$ (the linear hull of $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_n$).

1) Galerkin's approximations

For $n \in \mathbb{N}$, let us construct an approximation $\mathbf{u}_n$ in the form $\mathbf{u}_n(t) = \sum_{j=1}^{n} \alpha_j(t)\, \boldsymbol{\varphi}_j$ so that $\mathbf{u}_n$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{u}_n, \mathbf{w})_2 + \nu\big(A_\sigma \mathbf{u}_n, \mathbf{w}\big)_2 + \big\langle \mathcal{P}_\sigma \mathcal{B}(\mathbf{u}_n, \mathbf{u}_n), \mathbf{w}\big\rangle_{0,\sigma} \;=\; \langle \mathcal{P}_\sigma \mathbf{f}, \mathbf{w}\rangle_{0,\sigma} \tag{19}$$

for all $\mathbf{w} \in \boldsymbol{\mathcal{V}}_n$. This is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{u}_n, \boldsymbol{\varphi}_i)_2 + \nu\big(A_\sigma \mathbf{u}_n, \boldsymbol{\varphi}_i\big)_2 + \big\langle \mathcal{P}_\sigma \mathcal{B}(\mathbf{u}_n, \mathbf{u}_n), \boldsymbol{\varphi}_i\big\rangle_{0,\sigma} \;=\; \langle \mathcal{P}_\sigma \mathbf{f}, \boldsymbol{\varphi}_i\rangle_{0,\sigma}$$

for $i = 1, 2, \dots, n$. Using the ortho-normality of $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3, \dots$ and the identities $A_\sigma \mathbf{u}_n = \sum_{i=1}^{n} \alpha_i\, \boldsymbol{\varphi}_i = \sum_{i=1}^{n} \lambda_i \alpha_i\, \boldsymbol{\varphi}_i$, we obtain

$$\dot{\alpha}_i + \nu \lambda_i \alpha_i + \sum_{k,l=1}^{n} \alpha_k\, \alpha_l\, \big\langle \mathcal{P}_\sigma \mathcal{B}(\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_l), \boldsymbol{\varphi}_i\big\rangle_{0,\sigma} \;=\; \langle \mathcal{P}_\sigma \mathbf{f}, \boldsymbol{\varphi}_i\rangle_{0,\sigma} \qquad \text{for } i = 1, 2, \dots, n. \tag{20}$$

This is a system of n ODE's for the unknown coefficients $\alpha_1(t), \dots, \alpha_n(t)$. The system is solved with the initial conditions

$$\alpha_i(0) \;=\; (\mathbf{u}_0, \boldsymbol{\varphi}_i)_2 \qquad i = 1, \dots, n. \tag{21}$$

2) A priori estimates and existence of the Galerkin approximation $\mathbf{u}_n$

Multiply i–th equation by α_i and sum for $i = 1, \ldots, n$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{1}{2}\,\sum_{i=1}^{n}\alpha_i^2 + \nu\sum_{i=1}^{n}\lambda_i\alpha_i^2 \;&=\; \sum_{i=1}^{n}\alpha_i\big\langle\mathcal{P}_\sigma\mathbf{f},\boldsymbol{\varphi}_i\big\rangle_{0,\sigma} \;=\; \Big\langle\mathcal{P}_\sigma\mathbf{f},\sum_{i=1}^{n}\alpha_i\boldsymbol{\varphi}_i\Big\rangle_{0,\sigma}\\
&\leq\; \|\mathcal{P}_\sigma\mathbf{f}\|_{-1,2;\,\sigma}\,\Big\|\sum_{i=1}^{n}\alpha_i\boldsymbol{\varphi}_i\Big\|_{1,2} \;\leq\; c\,\|\mathbf{f}\|_{-1,2}\,\Big\|\nabla\sum_{i=1}^{n}\alpha_i\boldsymbol{\varphi}_i\Big\|_2\\
&=\; c\,\|\mathbf{f}\|_{-1,2}\left[\Big(\sum_{i=1}^{n}\alpha_i\nabla\boldsymbol{\varphi}_i,\sum_{j=1}^{n}\alpha_j\nabla\boldsymbol{\varphi}_j\Big)_2\right]^{\frac{1}{2}}\\
&=\; c\,\|\mathbf{f}\|_{-1,2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\langle A_\sigma\mathbf{u}_i,\mathbf{u}_j\rangle\right]^{\frac{1}{2}} \;=\; c\,\|\mathbf{f}\|_{-1,2}\left[\sum_{i=1}^{n}\alpha_i^2\lambda_i\right]^{\frac{1}{2}}\\
&\leq\; \frac{\nu}{2}\,\sum_{i=1}^{n}\lambda_i\alpha_i^2 + \frac{c}{\nu}\,\|\mathbf{f}\|_{-1,2}^2,
\end{aligned}
$$

where $c = c(\Omega)$. Integrating from 0 to t and multiplying by 2, we get

$$\sum_{i=1}^{n} \alpha_i^2(t) + \nu \int_0^t \sum_{i=1}^{n} \lambda_i \alpha_i^2(\tau)\, \mathrm{d}\tau \;\le\; c \int_0^t \|\mathbf{f}\|_{-1,2}^2\, \mathrm{d}\tau + \sum_{i=1}^{n} \alpha_i^2(0),$$

$$\sum_{i=1}^{n} \alpha_i^2(t) + \nu \int_0^t \sum_{i=1}^{n} \lambda_i \alpha_i^2(\tau)\, \mathrm{d}\tau \;\le\; C \int_0^t \|\mathbf{f}\|_{-1,2}^2\, \mathrm{d}\tau + \|\mathbf{u}_0\|_2^2, \tag{22}$$

$$\|\mathbf{u}_n(t)\|_2^2 + \nu \int_0^t \|\nabla \mathbf{u}_n(\tau)\|_2^2\, \mathrm{d}\tau \;\le\; c \int_0^t \|\mathbf{f}\|_{-1,2}^2\, \mathrm{d}\tau + \|\mathbf{u}_0\|_2^2. \tag{23}$$

One can deduce from these estimates that the initial–value problem (20), (21) has a solution $\alpha_1, \dots, \alpha_n$ on $(0,T)$. The solution satisfies inequality (22) for all $t \in (0,T)$. Hence the approximate solution $\mathbf{u}_n$ satisfies inequality (23) for all $t \in (0,T)$.

Note that returning to the first line on the previous page, we also obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \frac{1}{2}\, \|\mathbf{u}_n\|_2^2 + \nu\, \|\nabla \mathbf{u}_n\|_2^2 \;=\; \big\langle \mathcal{P}_\sigma \mathbf{f}, \mathbf{u}_n \big\rangle_{0,\sigma},$$

$$\|\mathbf{u}_n(t)\|_2^2 + 2\nu \int_0^t \|\nabla \mathbf{u}_n(\tau)\|_2^2\, \mathrm{d}\tau \;\le\; 2 \int_0^t \big\langle \mathcal{P}_\sigma \mathbf{f}, \mathbf{u}_n \big\rangle_{0,\sigma}\, \mathrm{d}\tau + \|\mathbf{u}_0\|_2^2. \tag{24}$$

3) Convergent subsequences of $\{\mathbf{u}_n\}$

Inequality (24) provides uniform estimates of $\mathbf{u}_n$ in $L^\infty(0,T;\ \mathbf{L}^2_\sigma(\Omega))$ and in $L^2(0,T;\ \mathbf{W}^{1,2}_{0,\sigma}(\Omega))$. Hence there exists a sub–sequence of $\{\mathbf{u}_n\}$ (denoted again by $\{\mathbf{u}_n\}$) and $\mathbf{u} \in L^\infty(0,T;\ \mathbf{L}^2_\sigma(\Omega)) \cap L^2(0,T;\ \mathbf{W}^{1,2}_{0,\sigma}(\Omega))$ such that

$$\mathbf{u}_n \longrightarrow \mathbf{u} \qquad \text{weakly–* in } L^\infty(0,T;\ \mathbf{L}^2_\sigma(\Omega)), \tag{25}$$

$$\mathbf{u}_n \longrightarrow \mathbf{u} \qquad \text{weakly in } L^2(0,T;\ \mathbf{W}^{1,2}_{0,\sigma}(\Omega)). \tag{26}$$

In order to proceed, we shall also need an information on a strong convergence of the sequence $\{\mathbf{u}_n\}$ in some space. Recall the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{u}_n,\mathbf{w})_2 + \nu\big(A_\sigma\mathbf{u}_n,\mathbf{w}\big)_2 + \big\langle \mathcal{P}_\sigma\mathcal{B}(\mathbf{u}_n,\mathbf{u}_n),\mathbf{w}\big\rangle_{0,\sigma} \;=\; \langle \mathcal{P}_\sigma\mathbf{f},\mathbf{w}\rangle_{0,\sigma} \tag{16}$$

for all $\mathbf{w} \in \boldsymbol{\mathcal{V}}_n$. As we already know that $\dot{\mathbf{u}}_n(t) \equiv \sum_{i=1}^n \dot{\alpha}_i(t)\,\boldsymbol{\varphi}_i$ exists, as a function from $(0,T)$ to $\boldsymbol{\mathcal{V}}_n$, at all points $t \in (0,T)$, we can also write equation (19) in the form

$$(\dot{\mathbf{u}}_n,\mathbf{w})_2 + \nu\big(A_\sigma\mathbf{u}_n,\mathbf{w}\big)_2 + \big\langle \mathcal{P}_\sigma\mathcal{B}(\mathbf{u}_n,\mathbf{u}_n),\mathbf{w}\big\rangle_{0,\sigma} \;=\; \langle \mathcal{P}_\sigma\mathbf{f},\mathbf{w}\rangle_{0,\sigma}. \tag{16}$$

Since $\boldsymbol{\mathcal{V}}_n \hookrightarrow \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \hookrightarrow \mathbf{L}^2_\sigma(\Omega) \hookrightarrow \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$, we may also treat $\dot{\mathbf{u}}_n$ as an element of $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. Its norm can be estimated:

$$
\begin{aligned}
\|\dot{\mathbf{u}}_n\|_{-1,2;\,\sigma} \;&=\; \sup_{\mathbf{w}\in\mathbf{W}^{1,2}_{0,\sigma}(\Omega),\ \mathbf{w}\neq\mathbf{0}} \frac{\big|\langle\dot{\mathbf{u}}_n,\mathbf{w}\rangle_{0,\sigma}\big|}{\|\mathbf{w}\|_{1,2}} \;=\; \sup_{\mathbf{w}\in\mathbf{W}^{1,2}_{0,\sigma}(\Omega),\ \mathbf{w}\neq\mathbf{0}} \frac{\big|(\dot{\mathbf{u}}_n,\mathbf{w})_2\big|}{\|\mathbf{w}\|_{1,2}} \\
&= \sup_{\mathbf{w}\in\boldsymbol{\mathcal{V}}_n,\ \mathbf{w}\neq\mathbf{0}} \frac{\big|(\dot{\mathbf{u}}_n,\mathbf{w})_2\big|}{\|\mathbf{w}\|_{1,2}} \;=\; \sup_{\mathbf{w}\in\boldsymbol{\mathcal{V}}_n,\ \mathbf{w}\neq\mathbf{0}} \frac{\big|\big\langle -A_\sigma\mathbf{u}_n-\mathcal{P}_\sigma\mathcal{B}(\mathbf{u}_n,\mathbf{u}_n)+\mathcal{P}_\sigma\mathbf{f},\mathbf{w}\big\rangle_{0,\sigma}\big|}{\|\mathbf{w}\|_{1,2}} \\
&\leq \|A_\sigma\mathbf{u}_n\|_{-1,2;\,\sigma}+\|\mathcal{P}_\sigma\mathcal{B}(\mathbf{u}_n,\mathbf{u}_n)\|_{-1,2;\,\sigma}+\|\mathcal{P}_\sigma\mathbf{f}\|_{-1,2;\,\sigma} \\
&= \|\mathcal{A}_\sigma\mathbf{u}_n\|_{-1,2;\,\sigma}+\|\mathcal{P}_\sigma\mathcal{B}(\mathbf{u}_n,\mathbf{u}_n)\|_{-1,2;\,\sigma}+\|\mathcal{P}_\sigma\mathbf{f}\|_{-1,2;\,\sigma} \\
&\leq \|\nabla\mathbf{u}_n\|_2+c\,\|\mathcal{B}(\mathbf{u}_n,\mathbf{u}_n)\|_{-1,2}+\|\mathbf{f}\|_{-1,2} \\
&\leq \|\nabla\mathbf{u}_n\|_2+c\,\|\nabla\mathbf{u}_n\|_2^{3/2}\,\|\mathbf{u}_n\|_2^{1/2}+c\,\|\mathbf{f}\|_{-1,2}.
\end{aligned}
$$

(The estimate of $\|\mathcal{B}(\mathbf{u}_n,\mathbf{u}_n)\|_{-1,2}$ holds due to (11).)

From this, we observe that the sequence $\{\dot{\mathbf{u}}_n\}$ is uniformly bounded in the space $L^{4/3}(0,T;\,\mathbf{W}^{-1,2}_{0,\sigma}(\Omega))$.

The next lemma is often called the Lions–Aubin lemma. (See e.g. Lions [7] or Temam [10].)

Lemma 3. *Let* $\mathbf{X}_0$, $\mathbf{X}$, $\mathbf{X}_1$ *be three Banach spaces such that* $\mathbf{X}_0$ *and* $\mathbf{X}_1$ *are reflexive and* $\mathbf{X}_0 \hookrightarrow\hookrightarrow \mathbf{X} \hookrightarrow \mathbf{X}_1$. *Let* $0 < T < \infty$, $1 < \alpha_1 < \infty$, $1 < \alpha_2 < \infty$. *Denote*

$$\mathcal{Y} \;:=\; \big\{\mathbf{z} \in L^{\alpha_0}(0,T;\,\mathbf{X}_0),\ \dot{\mathbf{z}} \in L^{\alpha_1}(0,T;\,\mathbf{X}_1)\big\}$$

the Banach space with the norm $\|z\|_{\mathcal{Y}} \;:=\; \|\mathbf{z}\|_{L^{\alpha_0}(0,T;\,\mathbf{X}_0)} + \|\dot{\mathbf{z}}\|_{L^{\alpha_1}(0,T;\,\mathbf{X}_1)}$.
Then $\mathcal{Y} \hookrightarrow\hookrightarrow L^{\alpha_0}(0,T;\,\mathbf{X})$ *(i.e. the injection of* $\mathcal{Y}$ *into* $L^{\alpha_0}(0,T;\,\mathbf{X})$ *is compact.*

We use the lemma with $X_0 = \mathbf{W}^{1,2}_{0,\sigma}(\Omega),\ X = \mathbf{L}^2_\sigma(\Omega),\ X_1 = \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$, $\alpha_0 = 2$, $\alpha_1 = \frac{4}{3}$.

As $\{\mathbf{u}_n\}$ is a bounded sequence in $\mathcal{Y}$, it is compact in $L^2(0,T;\,\mathbf{L}^2_\sigma(\Omega))$. Hence there exists a sub–sequence (denoted again $\{\mathbf{u}_n\}$) that, in addition to (25) and (26), satisfies

$$\mathbf{u}_n \;\longrightarrow\; \mathbf{u} \qquad \text{strongly in } L^2(0,T;\,\mathbf{L}^2_\sigma(\Omega)). \tag{27}$$

4) Verification that u satisfies equation (15)

Equation (19) means that

$$\int_0^T \int_\Omega \big[-\mathbf{u}_n \cdot \mathbf{w}\, \dot{\vartheta} + \nu \nabla \mathbf{u}_n : \nabla \mathbf{w}\, \vartheta + \mathbf{u}_n \cdot \nabla \mathbf{u}_n \cdot \mathbf{w}\, \vartheta \big]\, \mathrm{d}\mathbf{x}\, \mathrm{d}t$$
$$= \int_0^T \big\langle \mathcal{P}_\sigma \mathbf{f}, \mathbf{w} \big\rangle_{0,\sigma} \vartheta\, \mathrm{d}t + \vartheta(0) \int_\Omega \mathbf{u}_0 \cdot \mathbf{w}\, \mathrm{d}\mathbf{x} \tag{28}$$

for all $\mathbf{w} = \mathbf{w}(\mathbf{x}) \in \boldsymbol{\mathcal{V}}_n$ and all $\vartheta = \vartheta(t) \in C_0^\infty([0,T))$. Particularly, (28) also holds for all $\mathbf{w} \in \boldsymbol{\mathcal{V}}_m$, where $m \leq n$. Assume, for a while, that $\mathbf{w} \in \boldsymbol{\mathcal{V}}_m$ is fixed. Using all the types (25), (26), (27) of convergence of $\mathbf{u}_n$ to $\mathbf{u}$, one can pass to the limit (for $n \to \infty$) in (28) and show that

$$\int_0^T \int_\Omega \big[-\mathbf{u} \cdot \mathbf{w}\, \dot{\vartheta} + \nu \nabla \mathbf{u} : \nabla \mathbf{w}\, \vartheta + \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{w}\, \vartheta \big]\, \mathrm{d}\mathbf{x}\, \mathrm{d}t$$
$$= \int_0^T \big\langle \mathbf{f}, \mathbf{w} \big\rangle_{0,\sigma} \vartheta\, \mathrm{d}t + \vartheta(0) \int_\Omega \mathbf{u}_0 \cdot \mathbf{w}\, \mathrm{d}\mathbf{x} \tag{29}$$

for all $\mathbf{w} = \mathbf{w}(\mathbf{x}) \in \boldsymbol{\mathcal{V}}_m$ and all $\vartheta = \vartheta(t) \in C_0^\infty([0,T))$. Passing now to the limit for $m \to \infty$, we deduce that (29) holds for all $\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ and all functions ϑ. Now, it is equivalent to (15).

5) The energy inequality

Recall the inequality (24):

$$\|\mathbf{u}_n(t)\|_2^2 + 2\nu \int_0^t \|\nabla \mathbf{u}_n(\tau)\|_2^2 \, \mathrm{d}\tau \;\leq\; 2 \int_0^t \big\langle \mathcal{P}_\sigma \mathbf{f}, \mathbf{u}_n \big\rangle_{0,\sigma} \, \mathrm{d}\tau + \|\mathbf{u}_0\|_2^2.$$

The limit of the right hand side (for $n \to \infty$) is

$$= \; 2 \int_0^t \big\langle \mathcal{P}_\sigma \mathbf{f}(\tau), \mathbf{u} \big\rangle_{0,\sigma} \, \mathrm{d}\tau + \|\mathbf{u}_0\|_2^2 \; = \; 2 \int_0^t \big\langle \mathbf{f}(\tau), \mathbf{u} \big\rangle_0 \, \mathrm{d}\tau + \|\mathbf{u}_0\|_2^2.$$

The limit inferior of the left hand side (for $n \to \infty$) is

$$\geq \; \|\mathbf{u}(t)\|_2^2 + 2\nu \int_0^t \|\nabla \mathbf{u}_n(\tau)\|_2^2 \, \mathrm{d}\tau.$$

This yields the energy inequality

$$\|\mathbf{u}(t)\|_2^2 + 2\nu \int_0^t \|\nabla \mathbf{u}(\tau)\|_2^2 \, \mathrm{d}\tau \;\leq\; \|\mathbf{u}_0\|_2^2 + 2 \int_0^t \big\langle \mathbf{f}(\tau), \mathbf{u}(\tau) \big\rangle_0 \, \mathrm{d}\tau. \tag{17}$$

6) The strong right L^2–continuity of $\mathbf{u}$ at time $t = 0$

The energy inequality implies that

$$\limsup_{t\to 0+} \|\mathbf{u}(t)\|_2^2 \;\leq\; \|\mathbf{u}_0\|_2^2.$$

On the other hand, as $\mathbf{u}$ is weakly continuous from $[0, T)$ to $\mathbf{L}^2_\sigma(\Omega)$, we have

$$\liminf_{t\to 0+} \|\mathbf{u}(t)\|_2^2 \;\geq\; \|\mathbf{u}_0\|_2^2.$$

These inequalities yield

$$\lim_{t\to 0+} \|\mathbf{u}(t)\|_2^2 \;=\; \|\mathbf{u}_0\|_2^2.$$

This identity, together with the weak L^2–continuity, enable us to conclude that

$$\lim_{t\to 0+} \|\mathbf{u}(t) - \mathbf{u}_0\|_2^2 \;=\; 0.$$

It means that $\mathbf{u}(t) \to \mathbf{u}_0$ in $\mathbf{L}^2_\sigma(\Omega)$ for $t \to 0+$. $\blacksquare$

References

[1] H. Fujita, T. Kato: On the Navier-Stokes initial value problem. *Arch. Rat. Mech. Anal.* **16**, 1964, 269-315.

[2] G. P. Galdi: *An Introduction to the Mathematical Theory of the Navier–Stokes Equations.* 2nd edition, Springer 2011.

[3] G. P. Galdi: An Introduction to the Navier–Stokes initial–boundary value problem. In *Fundamental Directions in Mathematical Fluid Mechanics,* ed. G. P. Galdi, J. Heywood, R. Rannacher, series "Advances in Mathematical Fluid Mechanics". Birkhauser, Basel 2000, 1–98.

[4] E. Hopf: Uber die Anfganswertaufgabe für die Hydrodynamischen Grundgleichungen. *Math. Nachr.* **4**, 1951, 213–231.

[5] O. A. Ladyzhenskaya: *The Mathematical Theory of Viscous Incompressible Flow.* Gordon and Breach, New York 1963.

[6] J. Leray: Sur le Mouvements dun Liquide Visqueux Emplissant lEspace. *Acta Math.* **63**, 1934, 193–248.

[7] J. L. Lions: *Quelques Methodes de Resolution des Problemes aux Limites Non Lineaires.* Gauthier–Villars, Paris 1969.

[8] J. Neustupa: The role of pressure in the theory of weak solutions of the Navier–Stokes equations. In *Fluids under Pressure,* eds. T. Bodnár, G. P. Galdi and Š. Nečasová, to be published by Springer.

[9] H. Sohr: *The Navier–Stokes Equations. An Elementary Functional Analytic Approach.* Birkhäuser Advanced Texts, Basel–Boston–Berlin 2001.

[10] R. Temam: *Navier–Stokes Equations.* North–Holland, Amsterdam–New York–Oxford 1977.