An Introduction to Mathematical Modelling in Fluid Mechanics and Theory of the Navier-Stokes Equations

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Lecture 2:

The Stokes equations, the Stokes operator

(weak and strong solutions of the Stokes problem, basic properties of the Stokes operator)

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Contents

1. Some function spaces

Let Ω be a domain in $\mathbb{R}^3$. We denote vector functions and spaces of vector functions by boldface letters.

- $\mathbf{C}_{0,\sigma}^{\infty}(\Omega)$ ... the linear space of all infinitely differentiable divergence–free vector functions in Ω with a compact support in Ω,
- $\mathbf{L}_{\sigma}^{2}(\Omega)$ ... the closure of $\mathbf{C}_{0,\sigma}^{\infty}$ in $\mathbf{L}^2(\Omega)$,
- $\mathbf{W}_{0,\sigma}^{1,2}(\Omega) := \mathbf{W}_{0}^{1,2}(\Omega) \cap \mathbf{L}_{\sigma}^{2}(\Omega)$,
- $\mathbf{W}_{0,\sigma}^{-1,2}(\Omega)$ ... the dual to $\mathbf{W}_{0,\sigma}^{1,2}(\Omega)$,
- $\langle\, .\, ,\, .\, \rangle_{0,\sigma}$ ... the duality between elements of $\mathbf{W}_{0,\sigma}^{-1,2}(\Omega)$ and $\mathbf{W}_{0,\sigma}^{1,2}(\Omega)$.

Important properties of $\mathbf{L}_{\sigma}^{2}(\Omega)$. *1) Functions from $\mathbf{L}_{\sigma}^{2}(\Omega)$ have the divergence (in the sense of distributions) equal to zero in Ω.*

Proof. Let us denote by $\langle\!\langle\, .\, ,\, .\, \rangle\!\rangle_{\Omega}$ the action of a distribution in Ω on a function from $C_0^{\infty}(\Omega)$ or $\mathbf{C}_0^{\infty}(\Omega)$. Let $\mathbf{v} \in \mathbf{L}_{\sigma}^{2}(\Omega)$. Then there exists a sequence $\{\mathbf{v}_n\}$ in $\mathbf{C}_{0,\sigma}^{\infty}(\Omega)$, such that $\mathbf{v}_n \to \mathbf{v}$ (for $n \to \infty$) in the norm of $\mathbf{L}^2(\Omega)$. Let $\varphi \in C_0^\infty(\Omega)$. We have

$$\langle\!\langle \operatorname{div} \mathbf{v}, \varphi \rangle\!\rangle_\Omega = -\langle\!\langle \mathbf{v}, \nabla\varphi \rangle\!\rangle_\Omega = -\int_\Omega \mathbf{v} \cdot \nabla\varphi \, \mathrm{d}\mathbf{x} = -\lim_{n\to\infty} \int_\Omega \mathbf{v}_n \cdot \nabla\varphi \, \mathrm{d}\mathbf{x}$$
$$= \lim_{n\to\infty} \int_\Omega \operatorname{div} \mathbf{v}_n \, \varphi \, \mathrm{d}\mathbf{x} = 0. \qquad \blacksquare$$

2) Functions from $\mathbf{L}^2_\sigma(\Omega)$ have the normal component on $\partial\Omega$ equal to zero in a certain generalized sense of traces.

The explanation follows from the next lemma, see [2] or [3]:

Lemma 1. *Let Ω be a locally Lipschitz domain in $\mathbb{R}^3$ and $\mathbf{L}^2_{\mathrm{div}}(\Omega) := \{\mathbf{v} \in \mathbf{L}^2(\Omega);\ \operatorname{div} \mathbf{v} \in L^2(\Omega)\}$. There exists a continuous mapping $\gamma_n : \mathbf{L}^2_{\mathrm{div}}(\Omega) \to W^{-1/2,2}(\partial\Omega)$ such that $\gamma_n(\mathbf{v}) = \mathbf{v} \cdot \mathbf{n}\big|_{\partial\Omega}$ for $\mathbf{v} \in \mathbf{C}^\infty(\overline{\Omega})$.*

3) $\mathbf{L}^2_\sigma(\Omega)$ can be identified with a subspace of $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ so that for $\mathbf{v} \in \mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$, we put $\langle \mathbf{v}, \mathbf{w} \rangle_{0,\sigma} := (\mathbf{v}, \mathbf{w})_2$. Then $\mathbf{W}^{1,2}_{0,\sigma}(\Omega) \hookrightarrow \mathbf{L}^2_\sigma(\Omega) \hookrightarrow \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$.

The Helmholtz decomposition of $\mathrm{L}^2(\Omega)$.

Lemma 2. *Let Ω be any domain in $\mathbb{R}^3$. Then*

$$\mathbf{L}^2_\sigma(\Omega)^\perp \;=\; \mathbf{G}_2(\Omega) \;:=\; \big\{\mathbf{w} \in \mathbf{L}^2(\Omega);\; \mathbf{w} = \nabla\varphi \textit{ for some } \varphi \in W^{1,2}_{loc}(\Omega)\big\}.$$

Consequently,

$$\mathbf{L}^2(\Omega) \;=\; \mathbf{L}^2_\sigma(\Omega) \oplus \mathbf{G}_2(\Omega),$$

where $\mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{G}_2(\Omega)$ are closed orthogonal subspaces of $\mathbf{L}^2(\Omega)$.

Proof. Only the inclusion $\supset$. (See e.g. [2] or [3] or [6] for the opposite inclusion.) Thus, in order to show that $\mathbf{L}^2_\sigma(\Omega)^\perp \supset \mathbf{G}_2(\Omega)$, assume that $\nabla\varphi$ is an arbitrary element of $\mathbf{G}^2(\Omega)$. We want to show that $\nabla\varphi \in \mathbf{L}^2_\sigma(\Omega)^\perp$, which means that $(\mathbf{v}, \nabla\varphi)_2 = 0$ for all $\mathbf{v} \in \mathbf{L}^2_\sigma(\Omega)$. As $\mathbf{C}^\infty_{0,\sigma}(\Omega)$ is dense in $\mathbf{L}^2_\sigma(\Omega)$, it suffices to show that $(\mathbf{v}, \nabla\varphi)_2 = 0$ for all $\mathbf{v} \in \mathbf{C}^\infty_{0,\sigma}(\Omega)$. Thus, let $\mathbf{v} \in \mathbf{C}^\infty_{0,\sigma}(\Omega)$. Then we have

$$(\mathbf{v}, \nabla\varphi)_2 \;=\; \int_\Omega \mathbf{v}\cdot\nabla\varphi \,\mathrm{d}\mathbf{x} \;=\; \int_{\partial\Omega} (\mathbf{v}\cdot\mathbf{n})\,\varphi \,\mathrm{d}S - \int_\Omega (\mathrm{div}\,\mathbf{v})\,\varphi\,\mathrm{d}\mathbf{x} \;=\; 0. \qquad \blacksquare$$

Denote by P_σ, respectively Q_σ, the orthogonal projection of $\mathbf{L}^2(\Omega)$ onto $\mathbf{L}^2_\sigma(\Omega)$, respectively $\mathbf{G}_2(\Omega)$. Projection P_σ is often called the *Helmholtz projection.*

Remark. Let $\mathbf{v} \in \mathbf{L}^2(\Omega)$. The Helmholtz decomposition of $\mathbf{v}$ is: $\mathbf{v} = P_\sigma \mathbf{v} + \nabla\varphi$, where φ is a weak solution of the Neumann problem

$$\Delta\varphi = \operatorname{div}\mathbf{v} \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial\mathbf{n}} = \mathbf{v}\cdot\mathbf{n} \quad \text{on } \partial\Omega. \tag{1}$$

A weak solution: function $\varphi \in D^{1,2}(\Omega)$, satisfying

$$\int_\Omega \nabla\varphi\cdot\nabla\psi \, \mathrm{d}\mathbf{x} = \int_\Omega \mathbf{v}\cdot\nabla\psi \, \mathrm{d}\mathbf{x} \qquad \text{for all } \psi \in D^{1,2}(\Omega).$$

($D^{1,2}(\Omega)$ denotes the homogeneous Sobolev space $\{w \in L^1_{loc}(\Omega);\ \nabla w \in \mathbf{L}^2(\Omega)\}$ with the norm $\|w\|_{D^{1,2}(\Omega)} := \|\nabla w\|_2$.)

Remark. The analogous Helmholtz decomposition $\mathbf{L}^q(\Omega) = \mathbf{L}^q_\sigma(\Omega) \oplus \mathbf{G}_q(\Omega)$ (for $1 < q < \infty$) is possible $\Longleftrightarrow$ the Neumann problem (1) has a weak solution φ in $D^{1,q}(\Omega) := \{w \in L^1_{loc}(\Omega);\ \nabla w \in \mathbf{L}^q(\Omega)\}$ for any $\mathbf{v} \in \mathbf{L}^q(\Omega)$.

2. The Stokes problem

Let Ω be a domain in $\mathbb{R}^3$ and $T > 0$. We denote $Q_T := \Omega \times (0, T)$ and $\Gamma_T := \partial\Omega \times (0, T)$.

The non–steady Stokes initial–boundary value problem. Given functions $\mathbf{f}$ (in Q_T) and $\mathbf{u}_0$ (in Ω). The problem consists of the equations

$$\partial_t \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \mathbf{f} \qquad \text{in } Q_T, \tag{2}$$

$$\operatorname{div} \mathbf{u} = 0 \qquad \text{in } Q_T, \tag{3}$$

the boundary condition

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_T \tag{4}$$

and the initial condition

$$\mathbf{u} = \mathbf{u}_0 \qquad \text{in } \Omega \times \{0\}. \tag{5}$$

Equation (2) follows from the Navier–Stokes equation if we neglect the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$.

Here, we do not specially deal with the non–steady Stokes problem, because we shall discuss in greater detail the non–steady Navier–Stokes problem in next lectures.

The steady Stokes boundary value problem. Given function $\mathbf{f}$ (in Ω). The problem consists of the equations

$$-\nu\Delta\mathbf{u}+\nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{6}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{7}$$

and the boundary condition

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \partial\Omega. \tag{8}$$

We shall at first deal in greater detail with the steady problem (6)–(8).

A weak formulation of the steady Stokes problem (6)–(8). *Let* $\mathbf{f}\in\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. *Function* $\mathbf{u}\in\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ *is said to be a weak solution of the problem (6)–(8) if*

$$\nu\,(\nabla\mathbf{u},\nabla\mathbf{w})_2 \;\equiv\; \nu\int_\Omega \nabla\mathbf{u}:\nabla\mathbf{w}\,\mathrm{d}\mathbf{x} \;=\; \langle\mathbf{f},\mathbf{w}\rangle_{0,\sigma} \qquad \textit{for all } \mathbf{w}\in\mathbf{W}^{1,2}_{0,\sigma}(\Omega). \tag{9}$$

Note that (9) formally follows from (6)–(8) if we multiply equation (6) by $\mathbf{w}$ and integrate in Ω.

3. Operator $\mathcal{A}_\sigma$

Define a linear operator $\mathcal{A}_\sigma : \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \to \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ by the equation

$$\langle \mathcal{A}_\sigma \mathbf{v}, \mathbf{w} \rangle_{0,\sigma} \; = \; (\nabla \mathbf{v}, \nabla \mathbf{w})_2 \qquad \text{for all } \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega).$$

Now, we can write equivalently (9) in the form

$$\nu \mathcal{A}_\sigma \mathbf{u} \; = \; \mathbf{f}, \tag{10}$$

which is an equation in $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$.

Basic properties of operator $\mathcal{A}_\sigma$:

Lemma 3. $D(\mathcal{A}_\sigma) = \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$

(This follows directly from the definition of $\mathcal{A}_\sigma$.)

Lemma 4. *Operator $\mathcal{A}_\sigma$ is 1–1.*

Proof. Denote by $N(\mathcal{A}_\sigma)$ the null space of operator $\mathcal{A}_\sigma$. We need to show that $N(\mathcal{A}_\sigma) = \{\mathbf{0}\}$. Thus, let $\mathbf{v} \in N(\mathcal{A}_\sigma)$. Then

$$(\nabla \mathbf{v}, \nabla \mathbf{w})_2 \;=\; 0 \qquad \text{for all } \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega).$$

The choice $\mathbf{w} = \mathbf{v}$ yields: $\|\nabla \mathbf{v}\|_2 = 0$, which implies that $\mathbf{v} = \mathbf{0}$. ■

Lemma 5. *Operator $\mathcal{A}_\sigma$ is bounded.*

Proof. The boundedness of $\mathcal{A}_\sigma$ (as an operator from $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ follows from these identities and inequality:

$$\|\mathcal{A}_\sigma \mathbf{v}\|_{-1,2} \;=\; \sup_{\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega),\ \mathbf{w} \neq \mathbf{0}} \frac{|\langle \mathcal{A}_\sigma \mathbf{v}, \mathbf{w}\rangle_{0,\sigma}|}{\|\mathbf{w}\|_{1,2}} \;=\; \sup_{\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega),\ \mathbf{w} \neq \mathbf{0}} \frac{|(\nabla \mathbf{v},\, \nabla \mathbf{w})_2|}{\|\mathbf{w}\|_{1,2}} \;\leq\; \|\nabla \mathbf{v}\|_2.$$

■

Lemma 6. *The range of $\mathcal{A}_\sigma$ need not be generally the whole space $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$.*

Proof. Operator $\mathcal{A}_\sigma$ is closed because it is a bounded operator and its domain is the whole space $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. Hence $\mathcal{A}_\sigma^{-1}$ is also closed.

By contradiction: Assume that $R(\mathcal{A}_\sigma) \equiv D(\mathcal{A}_\sigma^{-1}) = \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. Then operator $\mathcal{A}_\sigma^{-1}$ is bounded (by the closed graph theorem).

Choose $\mathbf{z}_n \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ so that $\|\nabla \mathbf{z}_n\|_2 \to 0$ and $\|\mathbf{z}_n\|_2 \to 1$. (This choice is possible e.g. if Ω is an exterior domain or $\Omega = \mathbb{R}^3$.) Let $\mathbf{f}_n \in \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ be defined by the equation

$$\langle \mathbf{f}_n, \mathbf{w}\rangle_{0,\sigma} \ := \ (\nabla \mathbf{z}_n, \nabla \mathbf{w})_2 + (\mathbf{z}_n, \mathbf{w})_2 \qquad \text{for all } \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega).$$

Then $\{\mathbf{f}_n\}$ is a bounded sequence in $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. Put $\mathbf{u}_n := \mathcal{A}_\sigma^{-1}\mathbf{f}_n$. It means that $\mathbf{f}_n = \mathcal{A}_\sigma \mathbf{u}_n$. Hence

$$\langle \mathbf{f}_n, \mathbf{w}\rangle_{0,\sigma} \ := \ (\nabla \mathbf{u}_n, \nabla \mathbf{w})_2 \qquad \forall \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega).$$

The last two equations (with $\mathbf{w} = \mathbf{z}_n$ yield

$$(\nabla \mathbf{z}_n, \nabla \mathbf{z}_n)_2 + (\mathbf{z}_n, \mathbf{z}_n)_2 \ = \ (\nabla \mathbf{u}_n, \nabla \mathbf{z}_n)_2 \ \leq \ \|\nabla \mathbf{u}_n\|_2 \, \|\nabla \mathbf{z}_n\|_2$$

The left hand side tends to one, while the right hand side tends to zero (for $n \to \infty$). This is the contradiction. ■

Lemma 7. *The range of $\mathcal{A}_\sigma$ need not generally contain $\mathbf{L}^2_\sigma(\Omega)$.*

Proof. Assume, by contradiction, that $\mathbf{L}^2_\sigma(\Omega) \subset R(\mathcal{A}_\sigma)$. Denote by A_σ the restriction of $\mathcal{A}_\sigma$ to $\mathcal{A}_\sigma^{-1}(\mathbf{L}^2_\sigma(\Omega))$. In other words: $D(A_\sigma) = \{\mathbf{v} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega);\ \mathcal{A}_\sigma\mathbf{v} \in \mathbf{L}^2_\sigma(\Omega)\}$ and $A_\sigma\mathbf{v} = \mathcal{A}_\sigma\mathbf{v}$ for $\mathbf{v} \in D(A)$. Obviously, $R(A_\sigma) = \mathbf{L}^2_\sigma(\Omega)$.

Then $D(A_\sigma^{-1}) = \mathbf{L}^2_\sigma(\Omega)$ and $R(A_\sigma^{-1}) = D(A_\sigma) \subset \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$.

Let us at first show that A_σ^{-1} is closed, as an operator from $\mathbf{L}^2_\sigma(\Omega)$ to $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. In order to prove this, assume that $\{\mathbf{f}_n\}$ and $\{\mathbf{v}_n\}$ are sequences in $\mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$, respectively, such that $\mathbf{f}_n \to \mathbf{f}$ in $\mathbf{L}^2_\sigma(\Omega)$, $\mathbf{v}_n \to \mathbf{v}$ in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ and $\mathbf{v}_n = A_\sigma^{-1}\mathbf{f}_n$. We want to show that $\mathbf{v} = A_\sigma^{-1}\mathbf{f}$, i.e. $A_\sigma\mathbf{v} = \mathbf{f}$. However,

$$\mathbf{f}_n \to \mathbf{f} \quad \text{in } \mathbf{L}^2_\sigma(\Omega) \quad \Longrightarrow \quad \mathbf{f}_n \to \mathbf{f} \quad \text{in } \mathbf{W}^{-1,2}_{0,\sigma}(\Omega),$$

and we obtain $\mathbf{v} = \mathcal{A}_\sigma^{-1}\mathbf{f}$ (which equals $A_\sigma^{-1}\mathbf{f}$) due to the closedness of the operator $\mathcal{A}_\sigma^{-1}$ from $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$.

As a closed operator, defined on the whole space $\mathbf{L}^2_\sigma(\Omega)$, A_σ^{-1} is bounded from $\mathbf{L}^2_\sigma(\Omega)$ to $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ (by the closed graph theorem).

Choose $\{\mathbf{f}_n\}$ in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ so that $\|\mathbf{f}_n\|_2 \to 1$ and $\|\nabla\mathbf{f}_n\|_2 \to 0$. (A sequence with these properties exists e.g. if Ω is an exterior domain or $\Omega = \mathbb{R}^3$.)

Put $\mathbf{v}_n := A_\sigma^{-1}\mathbf{f}_n$. Then $A_\sigma\mathbf{v}_n = \mathbf{f}_n$, i.e. $\mathcal{A}_\sigma\mathbf{v}_n = \mathbf{f}_n$, which means:

$$(\nabla\mathbf{v}_n, \nabla\mathbf{w})_2 \;=\; \langle\mathbf{f}_n, \mathbf{w}\rangle_{0,\sigma} \;\equiv\; (\mathbf{f}_n, \mathbf{w})_2 \qquad \text{for all } \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega).$$

Using this with $\mathbf{w} = \mathbf{f}_n$, we get

$$\|\mathbf{f}_n\|_2^2 \;=\; (\nabla\mathbf{v}_n, \nabla\mathbf{f}_n)_2 \;\leq\; \|\nabla\mathbf{v}_n\|_2\,\|\nabla\mathbf{f}_n\|_2.$$

The left hand side tends to 1 (for $n \to \infty$).

The right hand side tends to 0, because $\|\nabla\mathbf{f}_n\|_2 \to 0$ and the sequence $\{\mathbf{v}_n\}$ is bounded in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ (because $\{\mathbf{f}_n\}$ is bounded in $\mathbf{L}^2_\sigma(\Omega)$ and operator A_σ^{-1} is bounded from $\mathbf{L}^2_\sigma(\Omega)$ to $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$).

This is a contradiction. ■

Domain and range of operator $\mathcal{A}_\sigma$:

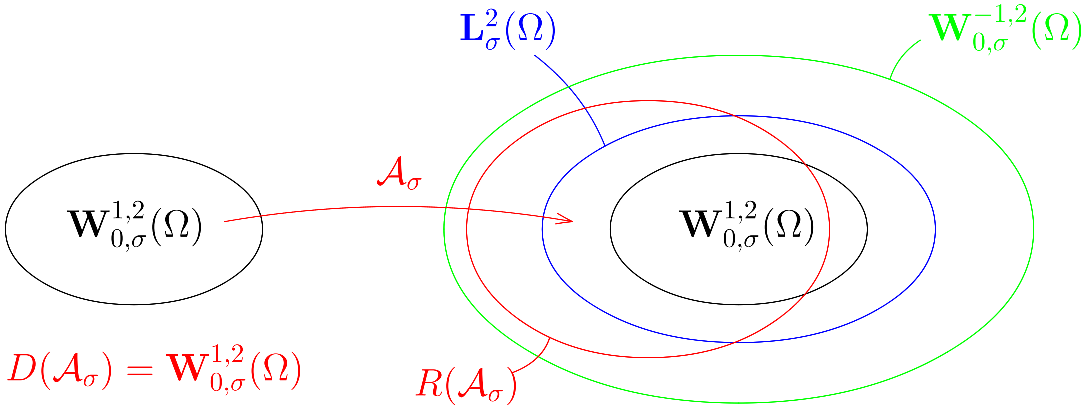

Fig. 1

The special case of
a bounded domain Ω:

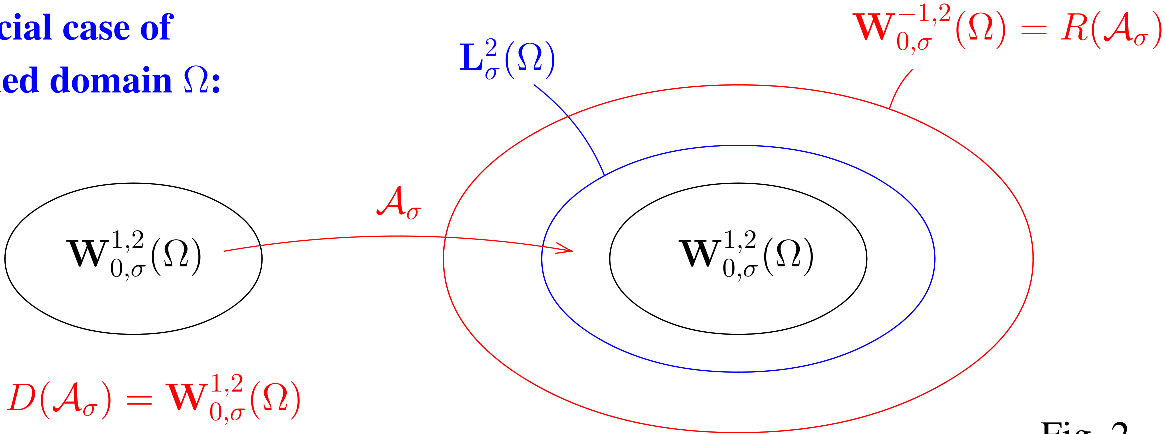

Fig. 2

Lemma 8. *If Ω is bounded then $R(\mathcal{A}_\sigma) = \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$.*

Proof. Since Ω is bounded, the scalar product $(\nabla \mathbf{v}, \nabla \mathbf{w})_2$ is equivalent to the scalar product $(\mathbf{v}, \mathbf{w})_{1,2}$ in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. Hence, given $\mathbf{f} \in \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$, there exists $\mathbf{v} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ such that $\langle \mathbf{f}, \mathbf{w} \rangle_{0,\sigma} = (\nabla \mathbf{v}, \nabla \mathbf{w})_2$ for all $\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ (by the Riesz theorem). It means that $\mathbf{f} = \mathcal{A}_\sigma \mathbf{v}$ (the identity in $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$). $\blacksquare$

Corollary 1. *If Ω is bounded then $\mathcal{A}_\sigma^{-1}$ is bounded from $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$.*

4. The Stokes operator A_σ

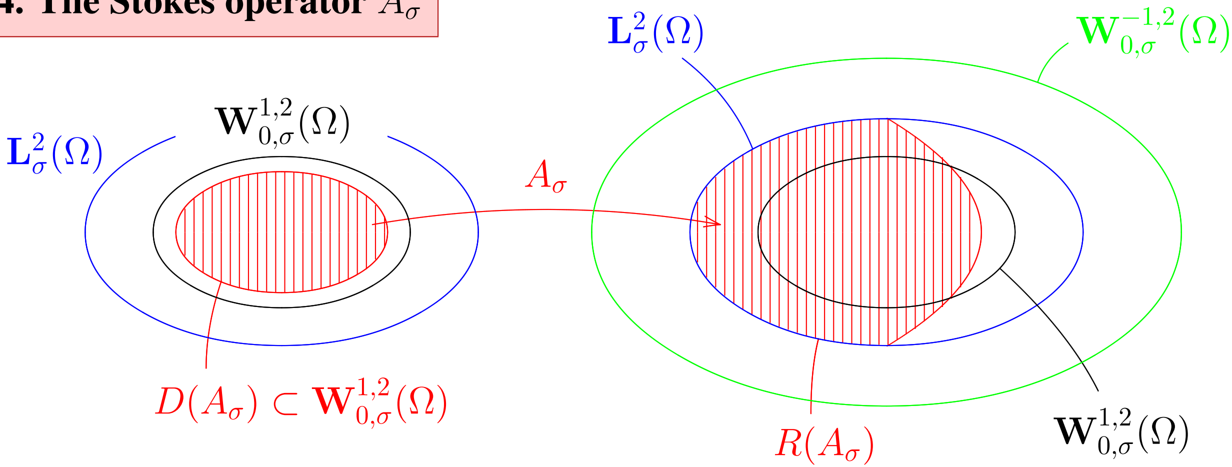

Fig. 3

Denote by A_σ the part of operator $\mathcal{A}_\sigma$ with the range $R(\mathcal{A}_\sigma) \cap \mathbf{L}^2_\sigma(\Omega)$. Thus, A_σ is the restriction of $\mathcal{A}_\sigma$ to

$$D(A_\sigma) := \left\{\mathbf{u} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega);\ \mathcal{A}_\sigma\mathbf{u} \in \mathbf{L}^2_\sigma(\Omega)\right\} = \mathcal{A}_\sigma^{-1}[R(\mathcal{A}_\sigma) \cap \mathbf{L}^2_\sigma(\Omega)].$$

Operator A_σ is an operator in $\mathbf{L}^2_\sigma(\Omega)$. It is often called the **Stokes operator.** We will treat it as an operator in $\mathbf{L}^2_\sigma(\Omega)$, i.e. the operator from $\mathbf{L}^2_\sigma(\Omega)$ to $\mathbf{L}^2_\sigma(\Omega)$, with the domain $D(A_\sigma)$ and range $R(A_\sigma)$, both subsets of $\mathbf{L}^2_\sigma(\Omega)$.

Some properties of operator A_σ: (see, e.g., [6])

Lemma 9. *A_σ is a 1–1 positive and self-adjoint operator in $\mathbf{L}^2_\sigma(\Omega)$.*

Proof. 1) A_σ is 1–1, because it is a restriction of $\mathcal{A}_\sigma$, which is 1–1.

2) Operator A_σ is positive, because for all $\mathbf{v} \in D(A_\sigma)$, $\mathbf{v} \neq \mathbf{0}$, we have

$$(A_\sigma \mathbf{v}, \mathbf{v})_2 \;=\; (\nabla \mathbf{v}, \nabla \mathbf{v})_2 \;=\; \|\nabla \mathbf{v}\|_2^2 \;>\; 0.$$

3) We prove that operator A_σ is closed. Let $\{\mathbf{v}_n\}$ and $\{\mathbf{f}_n\}$ be sequences in $D(A_\sigma)$ and $\mathbf{L}^2_\sigma(\Omega)$, respectively, such that $A_\sigma \mathbf{v}_n = \mathbf{f}_n$ and $\mathbf{v}_n \to \mathbf{v}$, $\mathbf{f}_n \to \mathbf{f}$ in the norm of $\mathbf{L}^2_\sigma(\Omega)$. In order to show that A_σ is closed, we need to show that $\mathbf{v} \in D(A_\sigma)$ and $A_\sigma \mathbf{v} = \mathbf{f}$. The equation $A_\sigma \mathbf{v}_n = \mathbf{f}_n$ means that

$$(\nabla \mathbf{v}_n, \nabla \mathbf{w})_2 \;=\; \langle \mathbf{f}_n, \mathbf{w} \rangle_{0,\sigma} \;=\; (\mathbf{f}_n, \mathbf{w})_2 \qquad \text{for all } \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega). \tag{11}$$

If $\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \cap \mathbf{W}^{2,2}(\Omega)$ the the left hand side equals

$$-(\mathbf{v}_n, \Delta \mathbf{w})_2 \;\longrightarrow\; -(\mathbf{v}, \Delta \mathbf{w})_2 \;=\; (\nabla \mathbf{v}, \nabla \mathbf{w})_2 \qquad \text{for } n \to \infty.$$

The right hand side of (11) tends to $(\mathbf{f}, \mathbf{w})_2$ (for $n \to \infty$). Hence

$$(\nabla \mathbf{v}, \nabla \mathbf{w})_2 \;=\; (\mathbf{f}, \mathbf{w})_2 \qquad \text{for all } \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \cap \mathbf{W}^{2,2}(\Omega).$$

As $\mathbf{W}^{1,2}_{0,\sigma}(\Omega) \cap \mathbf{W}^{2,2}(\Omega)$ is dense in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ the identity $(\nabla \mathbf{v}, \nabla \mathbf{w})_2 = (\mathbf{f}, \mathbf{w})_2$ holds for all $\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. This shows that $\mathbf{v} \in D(A_\sigma)$ and $A_\sigma \mathbf{v} = \mathbf{f}$. Thus, we have proven that operator A_σ is closed.

4) Let us now show that operator A_σ is symmetric: let $\mathbf{v}, \mathbf{w} \in D(A_\sigma)$ (which is dense in $\mathbf{L}^2_\sigma(\Omega)$). Then we have

$$(A_\sigma \mathbf{v}, \mathbf{w})_2 \;=\; (\nabla \mathbf{v}, \nabla \mathbf{w})_2 \;=\; (\nabla \mathbf{w}, \nabla \mathbf{v})_2 \;=\; (A_\sigma \mathbf{w}, \mathbf{v})_2 \;=\; (\mathbf{v}, A_\sigma \mathbf{w})_2.$$

5) Since A_σ is a closed, positive and symmetric operator in $\mathbf{L}^2_\sigma(\Omega)$, A_σ is self-adjoint. (See e.g. [4, Theorem V.3.16] or [1, Theorem 4.1.7].) ∎

Lemma 10. *The spectrum of A_σ is a closed subset of the real axis. Moreover, if domain Ω is bounded then the spectrum of A_σ consists of an increasing sequence of infinitely many isolated positive eigenvalues $\lambda_1, \lambda_2, \ldots,$ such that $\lim_{n\to\infty} \lambda_n = \infty$. Each of the eigenvalues has a finite geometric multiplicity and corresponding eigenfunctions can be chosen so that they form a complete orthonormal system in $\mathbf{L}^2_\sigma(\Omega)$.*

Proof. 1) The spectrum of A_σ is a closed subset of the real axis, because operator A_σ is symmetric.

2) We need to show that for each $\lambda \in \rho(A_\sigma)$ (the resolvent set of A_σ), the resolvent operator $(A_\sigma - \lambda I)^{-1}$ is compact in $\mathbf{L}^2_\sigma(\Omega)$. Then the statements

- the spectrum of A_σ consists of an increasing sequence of infinitely many isolated eigenvalues $\lambda_1,\ \lambda_2,\ \lambda_3,\ \dots,$
- each of the eigenvalues has a finite geometric multiplicity,
- corresponding eigenfunctions can be chosen so that they form a complete orthonormal system in $\mathbf{L}^2_\sigma(\Omega)$

follow from the spectral theorem for self-adjoint compact operators (or self-adjoint operators with compact resolvent), see e.g. [1] or [4] or many other books on the spectral theory of linear operators.

The eigenvalues are positive, because operator A_σ is positive. Their number is infinite, because the space $\mathbf{L}^2_\sigma(\Omega)$ is infinite–dimensional. The eigenvalues $\{\lambda_n\}$ satisfy $\lim_{n\to\infty} \lambda_n = \infty$ due to two reasons: a) operator A_σ is unbounded, or b) the eigenvalues cannot cluster at any point of $\mathbb{R}$, because they are isolated and the spectrum is closed.

Thus, let us show A_σ is an operator with a compact resolvent. Let $\lambda \in \rho(A_\sigma)$. Then $R(A_\sigma - \lambda I) = \mathbf{L}^2_\sigma(\Omega)$ and the operator $A_\sigma - \lambda I$ has a bounded inverse from $\mathbf{L}^2_\sigma(\Omega)$ to $\mathbf{L}^2_\sigma(\Omega)$. Let $\mathbf{f} \in \mathbf{L}^2_\sigma(\Omega)$. The equation

$$(A_\sigma - \lambda I)\mathbf{v} \;=\; \mathbf{f} \quad \ldots \quad \mathbf{v} \;=\; (A_\sigma - \lambda I)^{-1}\mathbf{f} \tag{12}$$

implies that

$$\|\mathbf{v}\|_2 \;=\; \|(A_\sigma - \lambda I)^{-1}\mathbf{f}\|_2 \;\le\; c\,\|\mathbf{f}\|_2, \tag{13}$$

where c depends only on λ. Equation (12) also means that

$$\begin{aligned}
&(\nabla\mathbf{v}, \nabla\mathbf{w})_2 \;=\; (\mathbf{f}, \mathbf{w})_2 + \lambda\,(\mathbf{v}, \mathbf{w})_2 \qquad \text{for all } \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega),\\
&\|\nabla\mathbf{v}\|_2^2 \;=\; (\mathbf{f}, \mathbf{v})_2 + \lambda\,(\mathbf{v}, \mathbf{v})_2 \;\le\; \|\mathbf{f}\|_2\,\|\mathbf{v}\|_2 + |\lambda|\,\|\mathbf{v}\|_2^2 \;\le\; c\,\|\mathbf{f}\|_2^2 + |\lambda|\,c^2\,\|\mathbf{f}\|_2^2.
\end{aligned}$$

This, together with (13), shows that $(A_\sigma - \lambda I)^{-1}$ is a bounded operator from $\mathbf{L}^2_\sigma(\Omega)$ to $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. Hence, due to the compact imbedding

$$\mathbf{W}^{1,2}_{0,\sigma}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^2_\sigma(\Omega),$$

the resolvent operator $(A_\sigma - \lambda I)^{-1}$ is compact in $\mathbf{L}^2_\sigma(\Omega)$. ■

Lemma 11. *If* $\mathbf{v} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \cap \mathbf{W}^{2,2}(\Omega)$ *then* $A_\sigma \mathbf{v} = -P_\sigma \Delta \mathbf{v}$.

Proof. For $\mathbf{v} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \cap \mathbf{W}^{2,2}(\Omega)$ and any $\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$, we have

$$(A_\sigma \mathbf{v}, \mathbf{w})_2 \;=\; (\nabla \mathbf{v}, \nabla \mathbf{w})_2 \;=\; -(\Delta \mathbf{v}, \mathbf{w})_2.$$

Since $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ is dense in $\mathbf{L}^2_\sigma(\Omega)$, we also have

$$(A_\sigma \mathbf{v} + \Delta \mathbf{v}, \mathbf{w})_2 \;=\; 0 \qquad \text{for all } \mathbf{w} \in \mathbf{L}^2_\sigma(\Omega).$$

This shows that $A_\sigma \mathbf{v} + \Delta \mathbf{v} \perp \mathbf{L}^2_\sigma(\Omega)$ in $\mathbf{L}^2(\Omega)$, which means that $A_\sigma \mathbf{v} + \Delta \mathbf{v} \in \mathbf{G}_2(\Omega)$. Hence $P_\sigma(A_\sigma \mathbf{v} + \Delta \mathbf{v}) = \mathbf{0}$. Since $P_\sigma A_\sigma \mathbf{v} = A_\sigma \mathbf{v}$, we obtain: $A_\sigma \mathbf{v} = -P_\sigma \Delta \mathbf{v}$. ■

Lemma 12. *If domain* Ω *is bounded then* $R(A_\sigma) \equiv D(A_\sigma^{-1}) = \mathbf{L}^2_\sigma(\Omega)$ *and operator* A_σ^{-1} *is bounded from* $\mathbf{L}^2_\sigma(\Omega)$ *to* $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$.

Proof. Since $R(\mathcal{A}_\sigma) = \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ (see Lemma 8) and $\mathbf{L}^2_\sigma(\Omega) \subset \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$, we have $R(A_\sigma) = \mathbf{L}^2_\sigma(\Omega)$. (See also Fig. 2.) Hence $D(A_\sigma^{-1}) = \mathbf{L}^2_\sigma(\Omega)$. We have shown in the proof of Lemma 7 that in this case, the operator A_σ^{-1} is closed, as an operator from $\mathbf{L}^2_\sigma(\Omega)$ to $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. The boundedness of A_σ^{-1} from $\mathbf{L}^2_\sigma(\Omega)$ to $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ now follows from the closed graph theorem.

Lemma 13. *If Ω is a bounded domain with the boundary of the class C^2 then $D(A_\sigma) = \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \cap \mathbf{W}^{2,2}(\Omega)$, $A_\sigma = -P_\sigma \Delta$ and*

$$\|\mathbf{u}\|_{2,2} \;\le\; c\, \|A_\sigma \mathbf{u}\|_2 \tag{14}$$

for all $\mathbf{u} \in D(A_\sigma)$.

Constant c in Lemma 13 depends only on domain Ω.

Lemma 13 shows that if Ω is a bounded "smooth" domain then operator A_σ has the so called **maximum regularity property.** This is a deep statement, see e.g. [2] or [5] or [6] for the proof.

5. More on the steady Stokes problem

A weak solution. Let $\mathbf{f} \in \mathbf{W}_{0,\sigma}^{-1,2})(\Omega)$. We already know from subsection 3 that the steady Stokes problem (6)–(8) is equivalent to the equation

$$\nu \mathcal{A}_\sigma \mathbf{u} \; = \; \mathbf{f} \tag{10}$$

in the space $\mathbf{W}_{0,\sigma}^{-1,2}(\Omega)$. A solution $\mathbf{u} \in \mathbf{W}_{0,\sigma}^{1,2}(\Omega)$ of equation (10) is called a *weak solution* (of the steady Stokes problem (6)–(8)).

- *If a solution exists then it is unique.* (See Lemma 4.)
- *However, equation (10) generally need not have a solution.* (See Lemma 6.)
- *On the other hand, if domain Ω is bounded then the solution $\mathbf{u} \in \mathbf{W}_{0,\sigma}^{1,2}(\Omega)$ exists for any $\mathbf{f} \in \mathbf{W}_{0,\sigma}^{-1,2}(\Omega)$.* (See Lemma 8.)

An associated pressure. Recall that equation (10) is equivalent to

$$\nu\,(\nabla\mathbf{u}, \nabla\mathbf{w})_2 \; \equiv \; \nu \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{w} \; \mathrm{d}\mathbf{x} \; = \; \langle \mathbf{f}, \mathbf{w} \rangle_{0,\sigma} \qquad \text{for all } \mathbf{w} \in \mathbf{W}_{0,\sigma}^{1,2}(\Omega). \tag{9}$$

Thus, if we consider only $\mathbf{w} \in \mathbf{C}^{\infty}_{0,\sigma}(\Omega)$ in (9), we can also write (9) in the form

$$\big\langle\!\big\langle \nu\Delta\mathbf{u} + \mathbf{f},\, \mathbf{w} \big\rangle\!\big\rangle_{\Omega} \;=\; 0.$$

This equation shows that the distribution $\nu\Delta\mathbf{u} + \mathbf{f}$ vanishes on all functions from $\mathbf{C}^{\infty}_{0}(\Omega)$ that are divergence–free. The next lemma tells us which form has such a distribution.

Lemma 14. *Let $\mathbf{F} = (F_1, F_2, F_3)$, where F_i $(i = 1, 2, 3)$ are distributions in Ω. Then $\mathbf{F}$ has the form $\mathbf{F} = \nabla p$ (where p is a distribution in Ω and ∇p is the distributional gradient) if and only if $\langle\!\langle \mathbf{F}, \mathbf{w} \rangle\!\rangle_{\Omega} = 0$ for all $\mathbf{w} \in \mathbf{C}^{\infty}_{0,\sigma}(\Omega)$.*

The lemma coincides with Proposition I.1.1 in [7]. It comes from G. De Rham.

Applying Lemma 14 with $\mathbf{F} = \nu\Delta\mathbf{u} + \mathbf{f}$, we deduce that there exists a distribution p in Ω such that

$$\nu\Delta\mathbf{u} + \mathbf{f} \;=\; \nabla p.$$

This equation formally coincides with the steady Stokes equation (6). Here, it is satisfied in the sense of distributions in Ω. Distribution p is called an *associated pressure.*

A strong solution. If $\mathbf{f} \in \mathbf{L}^2_\sigma(\Omega)$ then the steady Stokes problem is equivalent to the equation

$$\nu A_\sigma \mathbf{u} = \mathbf{f}, \tag{15}$$

which is now an equation in $\mathbf{L}^2_\sigma(\Omega)$. A solution of this equation is called a *strong solution.*

Concerning the existence and uniqueness of a strong solution, the situation is similar as in the case of a weak solution:

- *As $R(A_\sigma)$ generally does not cover the whole space $\mathbf{L}^2_\sigma(\Omega)$ (see Fig. 3), equation (15) is not always solvable.*
- *If a solution exists then it is unique.*

  (This follows from the facts that A_σ is a restriction of $\mathcal{A}_\sigma$ and operator $\mathcal{A}_\sigma$ is 1–1, see Lemma 4.)
- *If domain Ω is bounded then $R(A_\sigma) = \mathbf{L}^2_\sigma(\Omega)$, which means that equation (15) is solvable for any $\mathbf{f} \in \mathbf{L}^2_\sigma(\Omega)$.*

  (The identity $R(A_\sigma) = \mathbf{L}^2_\sigma(\Omega)$ follows from Lemma 8 and Fig. 2.)

- *If domain Ω is bounded and its boundary $\partial\Omega$ is of the class C^2 then the solution $\mathbf{u}$ of equation (15) lies in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega) \cap \mathbf{W}^{2,2}(\Omega)$. In this case, equation (15) can also be written in the form*

$$-\nu P_\sigma \Delta\mathbf{u} \;=\; \mathbf{f}. \tag{16}$$

Since $\mathbf{f}$ is now supposed to be from $\mathbf{L}^2_\sigma(\Omega)$, it satisfies $\mathbf{f} = P_\sigma\mathbf{f}$. Hence equation (16) is equivalent to

$$P_\sigma\big(\nu\Delta\mathbf{u} + \mathbf{f}\big) \;=\; \mathbf{0},$$

which is an equation in $\mathbf{L}^2_\sigma(\Omega)$. Due to the validity of the Helmholtz decomposition of $\mathbf{L}^2(\Omega)$ (see subsection 1), there exists $\nabla p \in \mathbf{G}_2(\Omega)$ such that

$$\nu\Delta\mathbf{u} + \mathbf{f} \;=\; \nabla p,$$

which is again the steady Stokes equation (6). Now, it is an equation in the space $\mathbf{L}^2(\Omega)$.

We observe, that under the formulated assumptions on $\mathbf{f}$ and Ω, the associated pressure p is a function from $L^1_{loc}(\Omega)$, such that $\nabla p \in \mathbf{L}^2(\Omega)$. The estimate (14) can be extended so that it also involves ∇p:

$$\|\mathbf{u}\|_{2,2} + \|\nabla p\|_2 \;\leq\; c\,\|\mathbf{f}\|_2. \tag{17}$$

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