An Introduction to Mathematical Modelling in Fluid Mechanics and Theory of the Navier-Stokes Equations

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Lecture 1:

An introduction to mathematical modelling in fluid mechanics

(from basic principles to the Navier–Stokes equations)

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Contents

1. A bit of history

Beginnings:

Aristoteles (384–322 B.C.), Archimedes (287–212 B.C.), Pascal (1623–1662), etc.

An important milestone: discovery of calculus (17th–18th century)

Newton (1642–1727), Bernoulli (1700–1782), Euler (1707–1783), etc.

18th–19th century:

Equations of motion of ideal and Newtonian fluids, first attempts to find solutions, simplified models. (Euler, Navier, Stokes, Prandtl, Reynolds and many others.)

20th–21st century:

Intensive analysis of the models, development, analysis and application of numerical methods, new models (fluids with more complicated rheologies).

2. Fluid and its kinematics, incompressible and compressible fluid, the rate of deformation tensor

The fluid is supposed to be a *continuum,* which means that the domain it occupies does not contain sets of Lebesgue measure zero and positive mass.

Eulerian description of fluid motion:

$\mathbf{u}(\mathbf{x},t)$, $\rho(\mathbf{x},t)$, $p(\mathbf{x},t)$, $\theta(\mathbf{x},t)$ $\ldots$ the velocity, density, pressure, temperature at point $\mathbf{x}$ and time t

Lagrangian description of fluid motion:

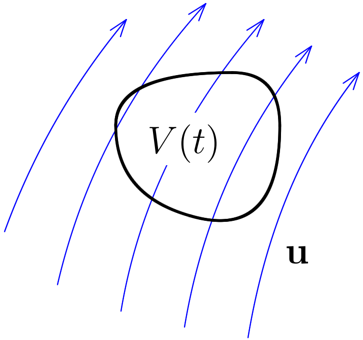
$\mathbf{X}(t;\, t_0,\mathbf{x}_0)$ $\ldots$ position (at time t) of a particle, whose position at time t_0 was $\mathbf{x}_0$

$$\mathbf{u}(\mathbf{x},t) \;=\; \frac{\partial}{\partial t}\mathbf{X}(t;\, t_0,\mathbf{x}_0), \quad \text{where } \mathbf{x}=\mathbf{X}(t;\, t_0,\mathbf{x}_0)$$

Lagrangian derivative with respect to t: $\quad D_t q(\mathbf{x},t) \;\equiv\; \dfrac{\mathrm{d}}{\mathrm{d}t}\, q(\mathbf{x},t)$

$$:=\; \frac{\mathrm{d}}{\mathrm{d}t}\, q\big(\mathbf{X}(t;\, t_0,\mathbf{x}_0),\, t\big) \;=\; \frac{\partial}{\partial t}\, q\big(\mathbf{X}(t;\, t_0,\mathbf{x}_0),\, t\big) + \mathbf{u}(\mathbf{x},t)\cdot\nabla q\big(\mathbf{X}(t;\, t_0,\mathbf{x}_0),\, t\big).$$

An important tool for further studies: Reynolds' transport formula



$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} q\,\mathrm{d}\mathbf{x} \;=\; \int_{V(t)}\left(\frac{\partial q}{\partial t} + \operatorname{div}(q\mathbf{u})\right)\mathrm{d}\mathbf{x}$$
$$= \int_{V(t)}\left(\frac{\partial q}{\partial t} + \frac{\partial}{\partial x_j}(q u_j)\right)\mathrm{d}\mathbf{x} \qquad (1)$$

A concrete motion of the fluid is called a flow.

A *flow* is said to be *incompressible* if all parts of the fluid preserve their volume.
Using Reynolds' formula with $q = 1$, we get

$$0 \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(V(t)) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \mathrm{d}\mathbf{x} \;=\; \int_{V(t)} \operatorname{div}\mathbf{u}\,\mathrm{d}\mathbf{x}.$$

From this, we observe that a *necessary and sufficient condition for flow* $\mathbf{u}$ *to be incompressible* is:

$$\operatorname{div}\mathbf{u} \;=\; 0. \qquad (2)$$

A fluid, whose each flow is incompressible, is said to be an *incompressible fluid.*

Tensor of deformation (the strain tensor):

$$\mathbb{S} \;:=\; \left(\nabla\mathbf{s}\right)_{\rm sym} + \frac{1}{2}\,(\nabla\mathbf{s})^T \cdot (\nabla\mathbf{s}) \;=\; \frac{1}{2}\left[\nabla\mathbf{s} + (\nabla\mathbf{s})^T + (\nabla\mathbf{s})^T \cdot (\nabla\mathbf{s})\right],$$

where $\mathbf{s} \equiv (s_1, s_2, s_3)$ is the vector of deformation and $\nabla\mathbf{s} = (\partial_i s_j)_{i,j=1,2,3}$.

Hence $\quad \mathbb{S} = (s_{ij})_{i,j=1,2,3} = \frac{1}{2}\left[\partial_i s_j + \partial_j s_i + (\partial_j s_i)(\partial_i s_j)\right]_{i,j=1,2,3}$.

Tensor of velocity of deformation (rate of deformation tensor, rate of strain tensor):

Vector of deformation of the fluid in a "short" time interval $[t, t+\tau]$: $\mathbf{s} = \mathbf{u}\tau$. The corresponding tensor of deformation depends on τ:

$$\mathbb{S}(\tau) \;=\; \left(\nabla\mathbf{u}\right)_{\rm sym}\tau + \frac{1}{2}\,(\nabla\mathbf{u})^T \cdot (\nabla\mathbf{u})\,\tau^2.$$

The derivative with respect to τ at the point $\tau = 0$ yields the *rate of deformation tensor:*

$$\mathbb{D} \;:=\; \left(\nabla\mathbf{u}\right)_{\rm sym} \;=\; \begin{pmatrix} \partial_1 u_1, & \frac{1}{2}(\partial_1 u_2 + \partial_2 u_1), & \frac{1}{2}(\partial_1 u_3 + \partial_3 u_1) \\ \frac{1}{2}(\partial_2 u_1 + \partial_1 u_2), & \partial_2 u_2, & \frac{1}{2}(\partial_2 u_3 + \partial_3 u_2) \\ \frac{1}{2}(\partial_3 u_1 + \partial_1 u_3), & \frac{1}{2}(\partial_3 u_2 + \partial_2 u_3), & \partial_3 u_3 \end{pmatrix}.$$

3. Acting forces, the stress tensor

The volume force: $$\mathbf{F}_{\text{vol}} := \int_{V(t)} \rho \mathbf{f} \,\mathrm{d}\mathbf{x},$$

where $\mathbf{f}$ is called the *specific volume force.* (It is the volume force, related to the unit of mass.)

The surface force: $$\mathbf{F}_{\text{surf}} := \int_{\partial V(t)} \mathbb{T} \cdot \mathbf{n} \,\mathrm{d}\mathbf{x},$$

where $\mathbb{T}$ is the so called *stress tensor* and $\mathbf{n}$ is the outer normal vector field. This expression of $\mathbf{F}_{\text{surf}}$ comes from A. Cauchy (1789–1857).

The structure of $\mathbb{T}$:

$$\mathbb{T} = \begin{pmatrix} \tau_{11}, & \tau_{12}, & \tau_{13} \\ \tau_{21}, & \tau_{22}, & \tau_{23} \\ \tau_{31}, & \tau_{32}, & \tau_{33} \end{pmatrix},$$

where τ_{ij} can be interpreted as the i–th component of the force with which the fluid acts on a unit planar surface, oriented by its normal vector in the direction of the x_j–axis.

Thus, the vector $\boldsymbol{\tau}_j := \begin{pmatrix} \tau_{1j} \\ \tau_{2j} \\ \tau_{3j} \end{pmatrix}$ (= the j–th column in tensor $\mathbb{T}$)

is the force with which the fluid acts on a unit planar surface, whose normal vector shows the direction of the x_j–axis.

A. Cauchy also proved that the stress tensor is symmetric, which means that $\tau_{ij} = \tau_{ji}$ for all $i, j = 1, 2, 3$. (Consequently, $\mathbb{T} = \mathbb{T}^T$.)

Finally, applying the Gauss divergence theorem, we obtain

$$\mathbf{F}_{\text{surf}} = \int_{V(t)} \operatorname{div} \mathbb{T} \, \mathrm{d}\mathbf{x} \quad \left(= \int_{V(t)} \left[\partial_1 \boldsymbol{\tau}_1 + \partial_2 \boldsymbol{\tau}_2 + \partial_3 \boldsymbol{\tau}_3 \right] \mathrm{d}\mathbf{x} \right).$$

4. Constitutive equations, Stokesian fluid

Constitutive equations: specify relations between the stress tensor and other quantities, typically (in the fluids) the dependence of the stress tensor and the rate of deformation tensor.

Stekesian fluid: the stress tensor satisfies these postulates:

a) the stress tensor $\mathbb{T}$ depends on velocity and its derivatives only through the rate of deformation tensor $\mathbb{D}$,

b) the stress tensor $\mathbb{T}$ does not explicitly depend on position $\mathbf{x}$ and time t,

c) the continuum is isotropic, i.e. it contains no preferred directions,

d) if the fluid is at rest then $\mathbb{T}$ is a multiple of the identity tensor $\mathbb{I}$ by a scalar.

Note that postulates a)–d) admit the explicit dependence of $\mathbb{T}$ on the so called *state quantities* ρ (density), p (pressure) and θ (temperature).

Tensor $\mathbb{T}$ may vary from point to point and from time to time. However due to postulate b), $\mathbb{T}$ depends on $\mathbf{x}$ and t only through $\mathbb{D}$, ρ, p and θ.

Postulate c) follows from a more general condition:

c') the way tensor $\mathbb{T}$ depends on tensor $\mathbb{D}$ is frame indifferent.

Lemma 1. *Postulate d) follows from postulates a), b), c').*

Proof. Denote by $\mathcal{F}(\mathbb{D})$ the dependence of $\mathbb{T}$ on $\mathbb{D}$: $\quad \mathbb{T} = \mathcal{F}(\mathbb{D})$.

If we identify $\mathbb{T}$ and $\mathbb{D}$ with 3×3 matrices in the orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and if we denote by $\mathbb{T}'$ and $\mathbb{D}'$ the 3×3 matrices which represent the same tensors in another orthonormal basis $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$, then

$$\mathbb{T}' = Q\, \mathbb{T}\, Q^T \qquad \text{and} \qquad \mathbb{D}' = Q\, \mathbb{D}\, Q^T,$$

where Q is the unitary matrix of transition from the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ to the basis $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$. Then

$$\mathbb{T}' = Q\, \mathcal{F}(\mathbb{D})\, Q^T.$$

Condition c') implies that $\mathbb{T}'$ depends on $\mathbb{D}'$ in the same way as $\mathbb{T}$ on $\mathbb{D}$, which means that $\mathbb{T}' = \mathcal{F}(\mathbb{D}') = \mathcal{F}(Q\,\mathbb{D}Q^T)$. Hence

$$\mathcal{F}(\mathbb{D}) \;=\; \mathbb{T} \;=\; Q^T\,\mathbb{T}'\,Q \;=\; Q^T\,\mathcal{F}(\mathbb{D}')\,Q \;=\; Q^T\,\mathcal{F}(Q\,\mathbb{D}\,Q^T)\,Q. \tag{3}$$

If the fluid is at rest then $\mathbb{D} = \mathbb{O}$ and (3) therefore yields

$$\mathcal{F}(\mathbb{O}) \;=\; Q^T\,\mathcal{F}(\mathbb{O})\,Q.$$

This identity holds for every unitary matrix Q. Hence $\mathcal{F}(\mathbb{O})$ is the so called *isotropic tensor.* Thus, $\mathcal{F}(\mathbb{O})$ is a scalar–multiple of the identity tensor. This verifies the fourth Stokes' postulate d).

The pressure. The dynamic stress tensor. Let the fluid be at rest. Since $\mathcal{F}(\mathbb{O})$ is a scalar multiple of tensor $\mathbb{I}$, there exists a scalar factor $-p$ such that $\mathcal{F}(\mathbb{O}) = -p\,\mathbb{I}$.

Calculating the traces of the terms in the equation $\mathbb{T} = -p\,\mathbb{I}$, we get

$$p \;=\; -\tfrac{1}{3}\,(\tau_{11} + \tau_{22} + \tau_{33}) \;=\; -\tfrac{1}{3}\,\mathrm{Tr}\,\mathbb{T}. \tag{4}$$

If we extend the hydrostatic concept of pressure to moving fluids then, however, formula (4) is not generally true. Nevertheless, in every moving Stokesian fluid, we have

$\mathbb{T} = \mathcal{F}(\mathbb{D}) = \mathcal{F}(\mathbb{O}) + [\mathcal{F}(\mathbb{D}) - \mathcal{F}(\mathbb{O})]$. This equation can be written in the form

$$\mathbb{T} \;=\; -p\,\mathbb{I} + \mathbb{T}_{\mathrm{d}} \tag{5}$$

where $\mathbb{T}_{\mathrm{d}} = \mathcal{F}(\mathbb{D}) - \mathcal{F}(\mathbb{O})$. The symmetric 2nd order tensor $\mathbb{T}_{\mathrm{d}}$ is called the *dynamic stress tensor.* It follows from (5) that

$$p \;=\; -\frac{1}{3}\,\mathrm{Tr}\,\mathbb{T} + \frac{1}{3}\,\mathrm{Tr}\,\mathbb{T}_{\mathrm{d}}.$$

G. R. Kirchhoff, coming from the molecular–kinetic theory of gases, showed that the appropriate formula, which generalizes (4), is

$$p \;=\; -\tfrac{1}{3}\,(\tau_{11} + \tau_{22} + \tau_{33}) + \mu'\,\mathrm{div}\,\mathbf{u} \;\equiv\; -\tfrac{1}{3}\,\mathrm{Tr}\,\mathbb{T} + \mu'\,\mathrm{div}\,\mathbf{u}. \tag{6}$$

μ' ... the *coefficient of volume viscosity,* or the *coefficient of the bulk viscosity.*

As a consequence of the 2nd law of thermodynamics, $\mu' \geq 0$.

If a considered fluid is incompressible then $\mathrm{div}\,\mathbf{u} = 0$ and (6) formally yields the same expression of p as (4).

Note that (5) and (6) yield

$$\mathrm{Tr}\,\mathbb{T}_{\mathrm{d}} \;=\; 3\mu'\,\mathrm{div}\,\mathbf{u}. \tag{7}$$

Theorem 1 (the general form of $\mathbb{T}_\mathrm{d}$ in a Stokesian fluid). *In a Stokesian fluid, the dynamic stress tensor $\mathbb{T}_\mathrm{d}$ has the general form*

$$\mathbb{T}_\mathrm{d} \;=\; \alpha\,\mathbb{I} + \beta\,\mathbb{D} + \gamma\,\mathbb{D}^2, \tag{8}$$

where the coefficients α, β and γ may depend on ρ, p, θ (density, pressure, temperature) and on the principal invariants I_1, I_2, I_3 of tensor $\mathbb{D}$.

Recall that a scalar function I, depending on tensor $\mathbb{D}$, is called an *invariant* of $\mathbb{D}$, if its value does not depend on the choice of an orthonormal basis, in which the tensor is expressed.

Examples. 1) The trace of a 2nd order tensor $\mathbb{A}$ is an invariant: let $\mathbb{A} = (a_{ij})$ be the representation of the tensor in the orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ in $\mathbb{R}^3$, let $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$ be another orthonormal basis and let $Q = (q_{ij})$ be a unitary matrix of transition from the first basis to the second basis. Then, if we denote by $\mathbb{A}' = (a'_{ij})$ the representation of the tensor in the second basis, we have

$$\mathbb{A}' = Q\,\mathbb{A}\,Q^T, \qquad \text{which means that} \quad a'_{ij} = q_{ik}\,a_{kl}\,q^T_{lj} = q_{ik}\,a_{kl}\,q_{jl}.$$

Hence

$$\operatorname{Tr}\mathbb{A}' \;=\; a'_{11}+a'_{22}+a'_{33} \;=\; a'_{ii} = q_{ik}\,a_{kl}\,q_{il} \;=\; \delta_{kl}\,a_{kl} \;=\; a_{kk} \;=\; \operatorname{Tr}\mathbb{A}.$$

We have shown that $\operatorname{Tr}\mathbb{A}$ is an invariant of tensor $\mathbb{A}$.

2) One can similarly show that if $\mathbb{A}$ is a second order tensor then $\operatorname{Tr}\mathbb{A}^2$, $\operatorname{Tr}\mathbb{A}^3$, etc., are also invariants of tensor $\mathbb{A}$.

3) Let $\mathbb{A}$ and $\mathbb{B}$ be 2nd order tensors, whose representations in the orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ in $\mathbb{R}^3$ are $\mathbb{A} = (a_{ij})$ and $\mathbb{B} = (b_{ij})$. Let $\mathbb{A}' = (a'_{ij})$ and $\mathbb{B}' = (b'_{ij})$ be representations of the same tensor in another orthonormal basis $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$ and let $Q = (q_{ij}$ be a unitary matrix of transition from the first basis to the second basis. Then

$$\mathbb{A}' : \mathbb{B}' \;=\; a'_{ij}\,b'_{ij} \;=\; q_{ik}\,q_{jl}\,a_{kl}\,q_{ir}\,q_{js}\,b_{rs} \;=\; \delta_{kr}\,\delta_{ls}\,a_{kl}\,b_{rs} \;=\; a_{kl}\,b_{kl} = \mathbb{A} : \mathbb{B}.$$

We have shown that the inner product $\mathbb{A} : \mathbb{B}$ is an invariant of both the tensors $\mathbb{A}$ and $\mathbb{B}$.

Eigenvalues, principal invariants. Recall that number λ is said to be an *eigenvalue* of the second order tensor $\mathbb{A}$ if there exists a non–zero vector $\mathbf{v}$ such that

$$(\mathbb{A} - \lambda \mathbb{I}) \cdot \mathbf{v} \;=\; \mathbf{0}. \tag{9}$$

It is well known that eigenvalues of tensor $\mathbb{A}$ are roots of the characteristic equation of $\mathbb{A}$: $\det(\mathbb{A} - \lambda \mathbb{I}) = 0$. This is a cubic equation for unknown λ, which can be written in the form

$$-\lambda^3 + I_1\,\lambda^2 - I_2\,\lambda + I_3 \;=\; 0.$$

One can calculate that

$$\begin{aligned}
I_1 &= \operatorname{Tr}\mathbb{A},\\
I_2 &= \frac{1}{2}\left[(\operatorname{Tr}\mathbb{A})^2 - \operatorname{Tr}(\mathbb{A}^2)\right],\\
I_3 &= \frac{1}{6}\left[(\operatorname{Tr}\mathbb{A})^3 - 3\,(\operatorname{Tr}\mathbb{A})\,(\operatorname{Tr}\mathbb{A}^2) + 2\,(\operatorname{Tr}\boldsymbol{\lambda}^3)\right].
\end{aligned}$$

As all the traces are invariants of tensor $\mathbb{A}$, I_1, I_2 and I_3 are also invariants. They are called the *principal invariants* of tensor $\mathbb{A}$.

Since all the coefficients in the characteristic equation are invariants of tensor $\mathbb{A}$, the eigenvalues are invariants of $\mathbb{A}$, too.

A scalar function of a 2nd order tensor is a mapping which assigns to each 2nd order tensor a number (the function value), independent of the representation of the tensor in a concrete orthonormal basis.

Lemma 2. *A scalar function of a symmetric 2nd order tensor $\mathbb{A}$ depends on $\mathbb{A}$ only through the principal invariants of $\mathbb{A}$.*

Proof. Let $\lambda_1,\ \lambda_2,\ \lambda_3$ be eigenvalues of $\mathbb{A}$. They are all real, because $\mathbb{A}$ is symmetric. The corresponding eigenvectors $\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$ can be chosen so that they form an orthonormal basis in $\mathbb{R}^3$. Since the scalar function of $\mathbb{A}$ is independent of the choice of an orthonormal basis in which $\mathbb{A}$ is represented, we may choose the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, in which the representation of $\mathbb{A}$ is

$$\begin{pmatrix} \lambda_1, & 0, & 0 \\ 0, & \lambda_2, & 0 \\ 0, & 0, & \lambda_3 \end{pmatrix}.$$

From this, we observe that the function depends on $\mathbb{A}$ only through the eigenvalues $\lambda_1,\ \lambda_2,\ \lambda_3$. However, they are uniquely given by the coefficients of the characteristic equation, i.e. by the principal invariants $I_1,\ I_2,\ I_3$. ∎

Proof of Theorem 1. We have:

$$\mathbb{T} \;=\; \mathcal{F}(\mathbb{D}) = \mathcal{F}(\mathbb{O}) + (\mathcal{F}(\mathbb{D}) - \mathcal{F}(\mathbb{O})) \;=\; -p\,\mathbb{I} + \mathcal{F}_{\mathrm{d}}(\mathbb{D}),$$

where $\mathcal{F}_{\mathrm{d}}(\mathbb{D}) = \mathcal{F}(\mathbb{D}) - \mathcal{F}(\mathbb{O})$ and $\mathbb{T}_{\mathrm{d}} = \mathcal{F}_{\mathrm{d}}(\mathbb{D})$.

We may identify $\mathbb{T}_{\mathrm{d}}$ and $\mathbb{D}$ with the representations in the orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Let Q be the matrix of transition from this basis to another orthonormal basis $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$. Then the representations of the same tensors in the second basis are

$$\mathbb{T}'_{\mathrm{d}} = Q\,\mathbb{T}_{\mathrm{d}}\,Q^T, \qquad \mathbb{D}' = Q\,\mathbb{D}\,Q^T.$$

Due to the principle of material frame indifference, we have: $\mathbb{T}'_{\mathrm{d}} = \mathcal{F}_{\mathrm{d}}(\mathbb{D}')$. Hence

$$Q\,\mathbb{T}_{\mathrm{d}}\,Q^T \;=\; \mathcal{F}_{\mathrm{d}}(Q\,\mathbb{D}\,Q^T), \quad \text{which yields} \quad \mathbb{T}_{\mathrm{d}} \;=\; Q^T\,\mathcal{F}_{\mathrm{d}}(Q\,\mathbb{D}\,Q^T)\,Q.$$

However, since $\mathbb{T}_{\mathrm{d}} = \mathcal{F}_{\mathrm{d}}(\mathbb{D})$, we get:

$$Q^T\,\mathcal{F}_{\mathrm{d}}(Q\,\mathbb{D}\,Q^T)\,Q \;=\; \mathcal{F}_{\mathrm{d}}(\mathbb{D}) \tag{10}$$

for every 3×3 unitary matrix Q.

Recall that tensor $\mathbb{D}$ is symmetric. Assume that $\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$ are the eigenvectors of $\mathbb{D}$. The representation of $\mathbb{D}$ in this basis is

$$\mathbb{D} \;=\; \begin{pmatrix} \lambda_1, & 0, & 0 \\ 0, & \lambda_2, & 0 \\ 0, & 0, & \lambda_3 \end{pmatrix},$$

where $\lambda_1,\ \lambda_2,\ \lambda_3$ are the (real) eigenvalues of $\mathbb{D}$. Consider the 3×3 unitary matrix

$$Q \;=\; \begin{pmatrix} -1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{pmatrix}.$$

Then $Q\,\mathbb{D}\,Q^T = \mathbb{D}$ and (10) yields: $Q^T\,\mathcal{F}_{\mathrm{d}}(\mathbb{D})\,Q = \mathcal{F}_{\mathrm{d}}(\mathbb{D})$.

Denote $\quad \mathcal{F}_{\mathrm{d}}(\mathbb{D}) = \begin{pmatrix} \xi_{11}, & \xi_{12}, & \xi_{13} \\ \xi_{21}, & \xi_{22}, & \xi_{23} \\ \xi_{31}, & \xi_{32}, & \xi_{33} \end{pmatrix}.$

We calculate: $\quad Q^T\,\mathcal{F}_{\mathrm{d}}(\mathbb{D})\,Q = \begin{pmatrix} \xi_{11}, & -\xi_{12}, & -\xi_{13} \\ -\xi_{21}, & \xi_{22}, & \xi_{23} \\ -\xi_{31}, & \xi_{32}, & \xi_{33} \end{pmatrix}.$

Hence $\xi_{12} = \xi_{21} = \xi_{13} = \xi_{31} = 0$.
Similarly, we can show that $\xi_{23} = \xi_{32} = 0$. Thus, we have

$$\mathbb{T}_{\mathrm{d}} = \mathcal{F}_{\mathrm{d}}(\mathbb{D}) = \begin{pmatrix} \xi_{11}, & 0, & 0 \\ 0, & \xi_{22}, & 0 \\ 0, & 0, & \xi_{33} \end{pmatrix}.$$

Suppose further, for simplicity, that $\lambda_1 \neq \lambda_2$, $\lambda_2 \neq \lambda_3$ and $\lambda_1 \neq \lambda_3$. Let (i, j, k) be a permutation of the numbers $(1, 2, 3)$. Then

$$\frac{(\mathbb{D} - \lambda_j \mathbb{I})(\mathbb{D} - \lambda_k \mathbb{I})}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)} \, \mathbf{e}_i = \mathbf{e}_i$$

and
$$\frac{(\mathbb{D} - \lambda_j \mathbb{I})(\mathbb{D} - \lambda_k \mathbb{I})}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)} \, \mathbf{e}_j = \frac{(\mathbb{D} - \lambda_j \mathbb{I})(\mathbb{D} - \lambda_k \mathbb{I})}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)} \, \mathbf{e}_k = \mathbf{0}.$$

It means that $\dfrac{(\mathbb{D} - \lambda_j \mathbb{I})(\mathbb{D} - \lambda_k \mathbb{I})}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)}$ is a projection onto the straight line, passing through the origin and oriented by vector $\mathbf{e}_i$.

Consequently, each vector $\mathbf{v} = (v_1, v_2, v_3) = v_1\,\mathbf{e}_1 + v_2\,\mathbf{e}_2 + v_3\,\mathbf{e}_3$ in $\mathbb{R}^3$ satisfies

$$\mathbf{v} \;=\; \underbrace{\frac{(\mathbb{D}-\lambda_2\mathbb{I})(\mathbb{D}-\lambda_3\mathbb{I})}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\,\mathbf{v}}_{v_1\,\mathbf{e}_1} + \underbrace{\frac{(\mathbb{D}-\lambda_1\mathbb{I})(\mathbb{D}-\lambda_3\mathbb{I})}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\,\mathbf{v}}_{v_2\,\mathbf{e}_2} + \underbrace{\frac{(\mathbb{D}-\lambda_1\mathbb{I})(\mathbb{D}-\lambda_2\mathbb{I})}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\,\mathbf{v}}_{v_3\,\mathbf{e}_3}$$

and $\quad \mathbb{T}_\mathrm{d}\,\mathbf{v} \;\equiv\; \xi_{11}\,v_1\,\mathbf{e}_1 + \xi_{22}\,v_2\,\mathbf{e}_2 + \xi_{33}\,v_3\,\mathbf{e}_3$

$$=\; \xi_{11}\,\frac{(\mathbb{D}-\lambda_2\mathbb{I})(\mathbb{D}-\lambda_3\mathbb{I})}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\,\mathbf{v} + \xi_{22}\,\frac{(\mathbb{D}-\lambda_1\mathbb{I})(\mathbb{D}-\lambda_3\mathbb{I})}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\,\mathbf{v} + \xi_{33}\,\frac{(\mathbb{D}-\lambda_1\mathbb{I})(\mathbb{D}-\lambda_2\mathbb{I})}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\,\mathbf{v}.$$

This yields $\quad \mathbb{T}_\mathrm{d}\,\mathbf{v} \;=\; \alpha\,\mathbb{I}\,\mathbf{v} + \beta\,\mathbb{D}\,\mathbf{v} + \gamma\,\mathbb{D}^2\,\mathbf{v} \;=\; (\alpha\,\mathbb{I} + \beta\,\mathbb{D} + \gamma\,\mathbb{D}^2)\,\mathbf{v},$

where the coefficients α, β, γ can be explicitly evaluated in terms of λ_1, λ_2, λ_3, ξ_{11}, ξ_{22} and ξ_{33}. The coefficients are scalar functions of the symmetric tensor $\mathbb{D}$. Thus, they depend on $\mathbb{D}$ only through the principal invariants I_1, I_2, I_3 of tensor $\mathbb{D}$. Naturally, they may also depend on ρ, p and θ.

Since $\mathbf{v}$ is an arbitrary vector in $\mathbb{R}^3$, we obtain formula (10).

5. Basic types of Stokesian fluids: ideal and Newtonian

Ideal fluid. The fluid is called *ideal* or *non–viscous* or *inviscid* or *perfect* if $\mathbb{T}_\mathrm{d} = \mathbb{O}$ and consequently, the stress tensor $\mathbb{T}$ has the form

$$\mathbb{T} \;=\; -p\mathbb{I}. \tag{11}$$

Newtonian fluid. The fluid is said to be *Newtonian* if the dynamic stress tensor $\mathbb{T}_\mathrm{d}$ depends linearly on the rate of deformation tensor $\mathbb{D}$. Then the coefficient γ in (8) is zero and (8) reduces to

$$\mathbb{T}_\mathrm{d} \;=\; \alpha\,\mathbb{I} + \beta\,\mathbb{D}, \tag{12}$$

where α and β may depend on the state variables ρ, p and θ. As $\mathbb{T}_\mathrm{d}$ should depend linearly on $\mathbb{D}$, β must be independent of $\mathbb{D}$ and α may depend linearly on $\mathbb{D}$ through its first invariant $I_1 \equiv \mathrm{Tr}\,\mathbb{D} \equiv \mathrm{div}\,\mathbf{u}$.

Calculating the traces of the tensors on both sides of (12), we obtain

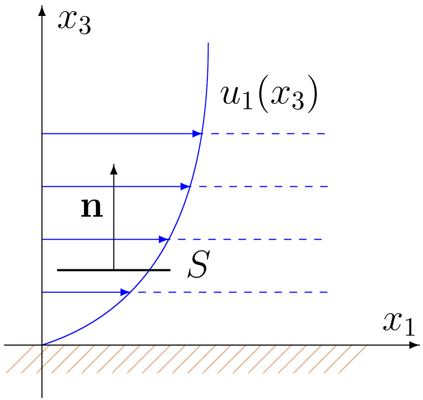
$$\mathrm{Tr}\,\mathbb{T}_\mathrm{d} \;=\; 3\alpha + \beta\,\mathrm{Tr}\,\mathbb{D} \;=\; 3\alpha + \beta\,\mathrm{div}\,\mathbf{u}.$$

Comparing this with
$$\operatorname{Tr} \mathbb{T}_{\mathrm{d}} = 3\mu' \operatorname{div} \mathbf{u}, \tag{7}$$

we obtain
$$3\mu' \operatorname{div} \mathbf{u} = 3\alpha + \beta \operatorname{div} \mathbf{u}.$$

Expressing α from this equation and substituting it back to (12), we get

$$\mathbb{T}_{\mathrm{d}} = \left[\left(\mu' - \tfrac{1}{3}\beta\right) \operatorname{div} \mathbf{u} \right] \mathbb{I} + \beta\, \mathbb{D}. \tag{13}$$



In order to assign a physical meaning to β, consider the flow on the picture. Here, $\mathbf{u} = (u_1, 0, 0)$, where $u_1 = u_1(x_3)$. Newton's law: the x_1–component of the force, acting onto a surface S, parallel with the x_1, x_2 –plane and oriented in the direction of the x_3–axis, related to the unit area of the surface, is proportional to $\mathrm{d}u_1/\mathrm{d}x_3$.

The coefficient of proportionality ... μ ... *dynamic coefficient of viscosity.* The considered x_1–component of the force coincides with the component τ_{13}^{d} of tensor $\mathbb{T}_{\mathrm{d}}$.

Thus, $\tau_{13}^{\mathrm{d}} = \mu \, \dfrac{\mathrm{d}u_1}{\mathrm{d}x_3}$.

From formula (13): $\tau_{13}^{\mathrm{d}} = \beta \, d_{13} = \beta \, \frac{1}{2} \left(\partial_3 u_1 + \partial_1 u_3 \right) = \dfrac{\beta}{2} \, \dfrac{\mathrm{d}u_1}{\mathrm{d}x_3}$.

Hence $\beta = 2\mu$. Formula (13) now yields:

$$\begin{aligned} \mathbb{T}_{\mathrm{d}} &= \left[(\mu' - \tfrac{2}{3}\mu) \operatorname{div} \mathbf{u} \right] \mathbb{I} + 2\mu\mathbb{D}, \\ \mathbb{T} &= \left[-p + (\mu' - \tfrac{2}{3}\mu) \operatorname{div} \mathbf{u} \right] \mathbb{I} + 2\mu\mathbb{D}. \end{aligned} \tag{14}$$

One can deduce by means of the 2nd law of thermodynamics that $\mu \geq 0$.

For incompressible fluids:

$$\mathbb{T}_{\mathrm{d}} = 2\mu\mathbb{D} \qquad \text{and} \qquad \mathbb{T} = -p \, \mathbb{I} + 2\mu\mathbb{D}. \tag{15}$$

6. Fundamental conservation laws, the Euler and Navier-Stokes equations for incompressible fluids

Fundamental laws:

1) The law of conservations of mass.
2) The law of conservation of momentum.
3) The law of conservation of energy.
4) The 2nd law of thermodynamics.

The laws 3) and 4) are especially important in the theory of compressible fluids.

As we deal only with incompressible fluids in this course, we explain only the use of laws 1) and 2) in this subsection.

The law of conservation of mass ($\rightarrow$ the equation of continuity). Let $\mathcal{P}$ be a part of the fluid, occupying domain $V(t)$ at time t. Since the mass of $\mathcal{P}$ does not depend on time, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho \,\mathrm{d}\mathbf{x} \;=\; 0.$$

Thus, due to Reynolds' formula (3) (which we use with $q = \rho$), we get

$$\int_{V(t)} \big[\partial_t \rho + \mathrm{div}\,(\rho \mathbf{u})\big] \,\mathrm{d}\mathbf{x} \;=\; 0.$$

Due to the freedom in the choice of the control sample $\mathcal{P}$ of the fluid, filling domain $V(t)$ at time t, we obtain the differential equation

$$\boxed{\partial_t \rho + \mathrm{div}\,(\rho \mathbf{u}) \;=\; 0}\,. \tag{16}$$

This is the *equation of continuity.* In an incompressible fluid, we have $\mathrm{div}\,\mathbf{u} = 0$, which implies: $\partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0$. It would be natural co call this equation "the equation of continuity for incompressible fluid". However, due to historical reasons, we call the *equation of continuity for incompressible fluid* the equation

$$\boxed{\mathrm{div}\,\mathbf{u} \;=\; 0}\,, \tag{2}$$

which coincides with the condition of incompressibility and which we obtain from (16) if we assume that ρ is a positive constant.

The law of conservation of momentum (→ Euler's and Navier–Stokes equations). Let $\mathcal{P}$ be a part of the fluid, occupying domain $V(t)$ at time t. Due to Euler's 1st law of mechanics (= the law of conservation of momentum), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho\mathbf{u}\;\mathrm{d}\mathbf{x} \;=\; \int_{V(t)} \rho\mathbf{f}\;\mathrm{d}\mathbf{x} + \int_{V(t)} \mathrm{div}\,\mathbb{T}\;\mathrm{d}\mathbf{x}.$$

The left hand side can be expressed by means of Reynolds' transport formula (3):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho\mathbf{u}\;\mathrm{d}\mathbf{x} \;=\; \int_{V(t)} \big[\partial_t(\rho\mathbf{u}) + \partial_j(\rho\mathbf{u}\,u_j)\big]\;\mathrm{d}\mathbf{x}.$$

Comparing the right hand sides and omitting the integrals, we obtain

$$\partial_t(\rho\mathbf{u}) + \partial_j(\rho u_j\mathbf{u}) \;=\; \rho\mathbf{f} + \mathrm{div}\,\mathbb{T}.$$

The left hand side can be modified by means of equation (16) so that we get:

$$\rho\,\partial_t\mathbf{u} + \rho u_j\,\partial_j\mathbf{u} \;=\; \rho\mathbf{f} + \mathrm{div}\,\mathbb{T},$$

$$\rho\,\partial_t\mathbf{u} + \rho\,\mathbf{u}\cdot\nabla\mathbf{u} \;=\; \rho\mathbf{f} + \mathrm{div}\,\mathbb{T}. \tag{17}$$

Each of these equations is a vectorial equation, which represents the system of three scalar equations

$$\rho\,\partial_t u_i + \rho u_j\,\partial_j u_i \;=\; \rho f_i + \partial_j \tau_{ij} \qquad (i = 1, 2, 3).$$

Euler's equation. If the fluid is ideal then $\mathbb{T} = -p\mathbb{I}$ and the equation of balance of momentum (17) takes the form

$$\boxed{\rho\,\partial_t \mathbf{u} + \rho\,\mathbf{u}\cdot\nabla\mathbf{u} \;=\; \rho\mathbf{f} - \nabla p}. \tag{18}$$

This is the so called *Euler equation.*

Navier–Stokes' equation. If the fluid is Newtonian then tensor $\mathbb{T}$ is given by formula (14) (in compressible case) or, particularly, by formula (15) (in the incompressible case). Thus, equation (17) takes the concrete form

$$\boxed{\rho\,\partial_t \mathbf{u} + \rho\,\mathbf{u}\cdot\nabla\mathbf{u} \;=\; \rho\mathbf{f} - \nabla p + \nabla\big[(\mu' - \tfrac{2}{3}\mu)\,\mathrm{div}\,\mathbf{u}\big] + \mathrm{div}\,[2\mu\mathbb{D}]} \tag{19}$$

(in the compressible case) or

$$\rho\,\partial_t \mathbf{u} + \rho\,\mathbf{u}\cdot\nabla\mathbf{u} \;=\; \rho\mathbf{f} - \nabla p + \mathrm{div}\,[2\mu\mathbb{D}]$$

(in the incompressible case).

If, in addition to the incompressibility, we assume that μ and ρ are positive constants, divide the equation by ρ and put $\nu := \mu/\rho$ (the so called *kinematic coefficient of viscosity*), we obtain

$$\boxed{\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} \ = \ \mathbf{f} - \frac{1}{\rho} \nabla p + \nu \Delta \mathbf{u}}. \tag{20}$$

Equation (19) is known as the *Navier–Stokes equation for compressible fluid* and equation (20) is known as the *Navier–Stokes equation for incompressible fluid.*

Equations of motion of an incompressible Newtonian fluid. Let us further focus on incompressible Newtonian fluids. The system of equations of motion consists of equations (2) (condition of incompressibility $\equiv$ equation of continuity for incompressible fluid) and (19) (the Navier–Stokes equation for incompressible fluid).

In order to simplify the equations, we usually put $\rho = 1$. (The system of physical units can always be chosen so that, in the chosen system, $\rho = 1$.)

Typical boundary conditions for velocity. Denote by Ω the domain in $\mathbb{R}^3$ where the fluid flows, and by $\mathbf{n}$ the outer normal vector field on $\partial\Omega$.

- for Euler's equation (non–viscous fluid): the *condition of impermeability:* $\mathbf{u}\cdot\mathbf{n}=0$ on $\partial\Omega$.
- for The Navier–Stokes equations (incompressible viscous Newtonian fluid):

  a) the *no–slip boundary condition:* $\mathbf{u}=\mathbf{0}$ on $\partial\Omega$,

  b) the *Navier's slip boundary conditions:* $\mathbf{u}\cdot\mathbf{n}=0$ and $[\mathbb{T}\cdot\mathbf{n}]_\tau+\gamma\mathbf{u}=\mathbf{0}$, where subscript τ denotes the tangential component and γ is the coefficient of friction between the fluid and the boundary.

  Note that the last condition can also be written: $[2\mu\,\mathbb{D}\cdot\mathbf{n}]_\tau+\gamma\mathbf{u}=\mathbf{0}$.

  One can show that if $\gamma=0$ (the so called *complete slip* or *perfect slip*) then the last condition reduces on flat parts of the boundary (where $\nabla\mathbf{n}=\mathbb{O}$) to $\mathbf{curl}\,\mathbf{u}\times\mathbf{n}=\mathbf{0}$. This condition is called the *Navier–type boundary condition.* It is often used (in conjunction with $\mathbf{u}\cdot\mathbf{n}=0$) on the whole boundary $\partial\Omega$.

We shall further focus on problems with the no–slip boundary condition $\mathbf{u}=\mathbf{0}$ on $\partial\Omega$ in this course.

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