On the Role of Pressure in Theory of the Navier–Stokes and MHD Equations II

Jiří Neustupa

Academy of Sciences of the Czech Republic

Institute of Mathematics

Yonsei University Seoul, June 2022

7. MHD equations and analogous notions and results for the MHD equations

MHD = Magneto–Hydro–Dynamics

The fluid is supposed to be electrically conductive.

$\mathbf{E}$	...	electric field	ϵ	...	permittivity
$\mathbf{b}$	...	magnetic field	μ	...	permeability
$\mathbf{u}$	...	velocity of motion of the fluid			
ρ_{e}	...	electric charge density	ρ_{f}	...	fluid density
$\mathbf{J}$	...	electric current density	σ	...	electric conductivity tensor

Maxwell's equations:

Gauss' law for the electric field:	$\operatorname{div}\mathbf{E} = \dfrac{\rho_\mathrm{e}}{\epsilon},$	(7.1)
Faraday's law of induction:	$\mathbf{curl}\ \mathbf{E} = -\partial_t \mathbf{b}$	(7.2)
Gauss' law for the magnetic field:	$\operatorname{div}\mathbf{b} = 0$	(7.3)
Ampère's law with Maxwell's addition:	$\mathbf{curl}\ \mathbf{b} = \mu \mathbf{J} + \epsilon\mu\, \partial_t \mathbf{E}$	(7.4)

Ohm's law:

$$\mathbf{J} = \sigma\,(\mathbf{E} + \mathbf{u} \times \mathbf{b}). \tag{7.5}$$

Transport equation for the charge density ρ_e:

Due to (7.4) and (7.1),

$$\begin{aligned} 0 &= \mu \operatorname{div}\mathbf{J} + \epsilon\mu\, \partial_t \operatorname{div}\mathbf{E} = \mu \operatorname{div}\mathbf{J} + \mu\, \partial_t \rho_\mathrm{e}, \\ \partial_t \rho_\mathrm{e} + \operatorname{div}\mathbf{J} &= 0. \qquad (7.6) \end{aligned}$$

Transport equation for b. Due to (7.2), we have

$$\begin{aligned}
\partial_t \mathbf{b} &= -\mathbf{curl}\ \mathbf{E} && \text{(Faraday'a law)} \\
&= -\mathbf{curl}\ (\sigma^{-1}\mathbf{J} - \mathbf{u}\times\mathbf{b}) && \text{(due to Ohm's law)} \\
&= -\mathbf{curl}\ \left[\sigma^{-1}\left(\mu^{-1}\,\mathbf{curl}\ \mathbf{b} - \epsilon\,\partial_t\mathbf{E}\right) - \mathbf{u}\times\mathbf{b}\right] && \text{(due to Ampère's law)} \\
&= \epsilon\,\partial_t\,\mathbf{curl}\ \mathbf{E} - \mathbf{curl}\ \left[\sigma^{-1}\,\mu^{-1}\,\mathbf{curl}\ \mathbf{b} - \mathbf{u}\times\mathbf{b}\right] && \\
&= -\epsilon\,\partial_t^2\mathbf{b} - \mathbf{curl}\ \left[\sigma^{-1}\,\mu^{-1}\,\mathbf{curl}\ \mathbf{b} - \mathbf{u}\times\mathbf{b}\right] && \text{(due to Faraday's law)}
\end{aligned}$$

Assuming further that $\sigma \in \mathbb{R}$, $\sigma > 0$ and using the formula $\mathbf{curl}^{\,2}\mathbf{b} = -\Delta\mathbf{b}$ (which holds for divergence–free vector fields), we get

$$\epsilon\,\partial_t^2\mathbf{b} + \partial_t\mathbf{b} + \mathbf{curl}\,(\mathbf{u}\times\mathbf{b}) \;=\; \frac{1}{\sigma\mu}\,\Delta\mathbf{b}$$

Neglecting the first term on the left hand side (due to the smallness of ϵ in comparison to other quantities), we finally obtain

$$\partial_t\mathbf{b} \;=\; \frac{1}{\mu\sigma}\,\Delta\mathbf{b} + \mathbf{curl}\ (\mathbf{u}\times\mathbf{b}). \tag{7.7}$$

The complete system of MHD equations for an incompressible Newtonian electrically conductive fluid.

We assume that the fluid is incompressible, electrically conductive, with a constant density.

The momentum equation with the specific Lorentz force:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} \;=\; \mathbf{f} - \frac{1}{\rho_{\mathrm{f}}} \nabla p + \nu \Delta \mathbf{u} + \mathbf{f}_L, \tag{7.8}$$

where $\mathbf{f}_L \;=\; \mathbf{J} \times \mathbf{b} \;=\; \Big(\dfrac{1}{\mu} \,\mathbf{curl}\; \mathbf{b} - \epsilon\, \partial_t \mathbf{E} \Big) \times \mathbf{b} \;\doteq\; \dfrac{1}{\mu}\, \mathbf{curl}\; \mathbf{b} \times \mathbf{b},$

the equation of continuity for the fluid:

$$\operatorname{div} \mathbf{u} \;=\; 0, \tag{7.9}$$

the transport equation for the magnetic field:

$$\partial_t \mathbf{b} \;=\; \frac{1}{\mu \sigma} \Delta \mathbf{b} + \mathbf{curl}\; (\mathbf{u} \times \mathbf{b}), \tag{7.10}$$

the Gauss law for the magnetic field:

$$\operatorname{div} \mathbf{b} \;=\; 0. \tag{7.11}$$

Transformation to the dimensionless form.

Denote by U, T,L, B, P and F characteristic units for the velocity, time, length, magnetic induction, pressure and the external body force respectively.

Then

$$\mathbf{u} = U\mathbf{u}', \quad t = Tt', \quad \mathbf{x} = L\mathbf{x}', \quad \mathbf{b} = B\mathbf{b}', \quad p = Pp', \quad \mathbf{f} = F\mathbf{f}',$$

where $\mathbf{u}'$, t', $\mathbf{x}'$, $\mathbf{b}'$, p' and $\mathbf{f}'$ represent the so called *dimensionless values* of the velocity, time, length, magnetic induction, pressure and the external body force, respectively.

Substituting to the equations (7.8)–(7.11) and omitting primes, we obtain

$$\begin{aligned}
\frac{U}{T}\,\partial_t\mathbf{u} + \frac{U^2}{L}\,\mathbf{u}\cdot\nabla\mathbf{u} \;&=\; F\,\mathbf{f} - \frac{P}{\rho_{\mathrm{f}}L}\,\nabla p + \frac{\nu U}{L^2}\,\Delta\mathbf{u} + \frac{B^2}{\mu L}\,\mathbf{curl}\,\mathbf{b}\times\mathbf{b}, \\
\frac{U}{L}\,\mathrm{div}\,\mathbf{u} \;&=\; 0, \\
\frac{B}{T}\,\partial_t\mathbf{b} + \frac{UB}{L}\,\mathbf{curl}\;(\mathbf{b}\times\mathbf{u}) \;&=\; \frac{B}{\mu\sigma L^2}\,\Delta\mathbf{b}, \\
\frac{B}{L}\,\mathrm{div}\,\mathbf{b} \;&=\; 0.
\end{aligned}$$

Choosing $T := L/U$, $P := \rho_{\mathrm{f}} U^2$, $F = U^2/L$, $B = \sqrt{\mu} U$ and multiplying the first equation by L/U^2, the second equation by L/U, the third equation by $L/(UB)$ and the fourth equation by L/B, we obtain the system

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{f} - \nabla p + \frac{1}{\mathcal{R}_f} \Delta \mathbf{u} + \mathbf{curl}\, \mathbf{b} \times \mathbf{b}, \tag{7.12}$$

$$\operatorname{div} \mathbf{u} = 0, \tag{7.13}$$

$$\partial_t \mathbf{b} + \mathbf{curl}\, (\mathbf{b} \times \mathbf{u}) = \frac{1}{\mathcal{R}_m} \Delta \mathbf{b}, \tag{7.14}$$

$$\operatorname{div} \mathbf{b} = 0, \tag{7.15}$$

where $\mathcal{R}_f := LU/\nu$ is the *fluid Reynolds number* and $\mathcal{R}_m := \mu\sigma LU$ is the *magnetic Reynolds number.*

Note that the equations (7.12) and (7.14) can also be written in the form

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{f} - \nabla\big(p + \tfrac{1}{2}|\mathbf{b}|^2\big) + \frac{1}{\mathcal{R}_f} \Delta \mathbf{u} + \mathbf{b} \cdot \nabla \mathbf{b}, \tag{7.12}$$

$$\partial_t \mathbf{b} + \mathbf{u} \cdot \nabla \mathbf{b} = \frac{1}{\mathcal{R}_m} \Delta \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{u}. \tag{7.14}$$

8. An interior regularity of the pressure in regions, where the velocity satisfies Serrin's integrability condition

Motivation. Let the initial–boundary value problem for the Navier–Stokes or MHD equations in $Q_T := \Omega \times (0,T)$ be given. It often happens that we study properties of solutions only in a sub-domain $\Omega' \times (T_1,T_2)$, where $\Omega' \subset\subset \Omega$ and $0 \leq T_1 < T_2 \leq T$. In order to reduce the problem in Q_T just to the problem in $\Omega' \times (T_1,T_2)$, we can use the *method of localization.*

Consider, for simplicity, just the Navier–Stokes equations. (I.e. $\mathbf{b} = \mathbf{0}$ in the MHD system.)

Let $\Omega_2 \subset\subset \Omega_1 \subset \Omega$ and let η be an infinitely differentiable function in $\mathbb{R}^3$, such that

$$\eta \begin{cases} = 1 & \text{in } \Omega_2, \\ \in [0,1] & \text{in } \Omega_1 \smallsetminus \Omega_2, \\ = 0 & \text{in } \mathbb{R}^3 \smallsetminus \Omega_1. \end{cases}$$

Multiply $\mathbf{u}$ and p by η. The function $\eta\mathbf{u}$, however, is not divergence–free: $\operatorname{div}(\eta\mathbf{u}) = \nabla\eta \cdot \mathbf{u}$. Thus, we find $\mathbf{U}$ such that $\operatorname{div}\mathbf{U} = \nabla\eta \cdot \mathbf{u}$ and put $\widehat{\mathbf{u}} := \eta\mathbf{u} - \mathbf{U}$. The function

$\mathbf{U}$ can be constructed by means of the so called Bogovskij–Pileckas operator. One can derive that $\widehat{\mathbf{u}}$ and $\widehat{p} := \eta p$ satisfy

$$
\begin{aligned}
\partial_t \widehat{\mathbf{u}} + \widehat{\mathbf{u}} \cdot \nabla \widehat{\mathbf{u}} + \nabla \widehat{p} &= \nu \Delta \widehat{\mathbf{u}} + \mathbf{g}, && (8.16) \\
\operatorname{div} \widehat{\mathbf{u}} &= 0 && (8.17)
\end{aligned}
$$

in $\mathbb{R}^3 \times (\delta, T-\delta)$, where

$$
\begin{aligned}
\mathbf{g} &= \eta \mathbf{f} - \partial_t \mathbf{U} - \mathbf{U} \cdot \nabla(\eta \mathbf{u}) - (\eta \mathbf{u}) \cdot \nabla \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} + (\eta \mathbf{u} \cdot \nabla \eta)\, \mathbf{u} \\
&\quad - \eta\,(1-\eta)\, \mathbf{u} \cdot \nabla \mathbf{u} - 2\nu\, \nabla \eta \cdot \nabla \mathbf{u} - \nu\, \mathbf{u}\, \Delta \eta + \nu \Delta \mathbf{U} + p\, \nabla \eta. && (8.18)
\end{aligned}
$$

The function $\mathbf{g}$ depends on $\mathbf{U}$, p and η. We would like to have $\mathbf{g}$ as "nice" as possible.

If $(\mathbf{u}, p)$ is a suitable weak solution then the set of its singular points has the 1D Hausdorff measure equal to 0.

$(\mathbf{x}_0, t_0) \in Q_T$ is said to be a *regular point* of the solution $(\mathbf{u}, p)$ if $\mathbf{u}$ is essentially bounded in some space–time neighborhood of $(\mathbf{x}_0, t_0)$.

Each point in Q_T that is not regular, is said to be a *singular point* of the solution $(\mathbf{u}, p)$.

Denote by $\mathcal{S}[\mathbf{u}, p]$ the set of singular points.

Due to the information on the 1D Hausdorff measure of $\mathcal{S}[\mathbf{u},p]$, Ω_1 and Ω_2 can be chosen so that $[\overline{\Omega_1 \smallsetminus \Omega_2} \times (0,T)] \cap \mathcal{S}[\mathbf{u},p] = \emptyset$ and $\partial\Omega_1$ is arbitrarily close to $\partial\Omega'$.

Important question:

How regular are u and p in $(\Omega_1 \smallsetminus \Omega_2) \times (T_1, T_2)$?

Answers:

1) $\mathbf{u}$ is essentially bounded in $(\Omega_1 \smallsetminus \Omega_2) \times (\delta, T-\delta)$ for any $\delta > 0$, $\delta < T$. (Follows from the definition of a regular point.)

2) $\mathbf{u}$ and all its spatial partial derivatives of all orders are essentially bounded in $(\Omega_1 \smallsetminus \Omega_2) \times (T_1+\delta, T_2-\delta)$ for any $\delta > 0$, $\delta < T_2 - T_1$. (Follows from the result by J. Serrin [9], which says:

> *If* $\Omega' \subset \Omega$, $(T_1, T_2) \subset (0,T)$ *and* $\mathbf{u} \in L^r(T_1, T_2; \mathbf{L}^s(\Omega'))$, *where* $2/r + 3/s = 1$, $2 < r < \infty$, *then* $\mathbf{u}$ *and all its spatial partial derivatives of all orders are essentially bounded in* $\Omega'' \times (T_1+\delta, T_2-\delta)$ *for any domain* $\Omega'' \subset\subset \Omega'$ *and* $0 < \delta < T_2 - T_1$.

3) $\mathbf{u}$ is Hölder–continuous in $(\Omega_1 \smallsetminus \Omega_2) \times (T_1+\delta, T_2-\delta)$ for any $\delta > 0$, $\delta < T_2 - T_1$. (Follows from the result by A. Mahalov, B. Nicolaenko and T. Shilkin [3].

Similar results also hold for the MHD equations. (See [3].) Here, it follows from equation (7.14) that one also can make the same statement on $\partial_t \mathbf{b}$ as on $\mathbf{u}$ and $\mathbf{b}$.

These results, however, say nothing about the regularity of p and $\partial_t \mathbf{u}$.

Interior regularity of p and $\partial_t \mathrm{u}$ in regions where u satisfies Serrin's condition.

Important condition (Serrin's condition):

(i) $\mathbf{u} \in L^{\alpha}(T_1, T_2;\, \mathbf{L}^{\beta}(\Omega'))$ for some $\alpha, \beta \in \mathbb{R}$ such that $\dfrac{2}{\alpha} + \dfrac{3}{\beta} = 1,\ 3 < \beta < \infty$.

The Navier–Stokes equations in $\mathbb{R}^3$:

Theorem 1. *Let $\Omega = \mathbb{R}^3$. Let Ω' be a domain in $\mathbb{R}^3$, $0 \leq T_1 < T_2 \leq T$ and let $\mathbf{u}$ be a weak solution to the Navier-Stokes system (with $\mathbf{f} = \mathbf{0}$) in $\mathbb{R}^3 \times (0, T)$, satisfying condition (i) in $\Omega' \times (T_1, T_2)$. Let p be an associated pressure. Then $\partial_t \mathbf{u}$, ∇p and all their spatial derivatives (of all orders) are in $\mathbf{L}^{\infty}(\Omega'' \times (T_1 + \delta, T_2 + \delta))$ for any $\delta > 0$, $T_1 + \delta < T_2 - \delta$ and $\Omega'' \subset\subset \Omega'$.*

Since $p \in L^{5/3}(\mathbb{R}^3 \times (0,T))$ and ∇p has all spatial derivatives in $\mathbf{L}^\infty(\Omega'' \times (T_1+\delta, T_1-\delta))$ due to Theorem 1, the function $\vartheta \in L^{5/3}(0,T)$ can be chosen so that $p+\vartheta \in L^\infty(\Omega'' \times (T_1+\delta, T_2-\delta))$. (It is e.g. sufficient to put $\vartheta(t) := \int_{\Omega''} p(\mathbf{x},t)\ \mathrm{d}\mathbf{x}$.) Then all spatial derivatives of $(p+\vartheta)\,\nabla\eta$ are in $\mathbf{L}^\infty(\Omega'' \times (T_1+\delta, T_2-\delta))$.

The Navier–Stokes equations, the case $\Omega \neq \mathbb{R}^3$, no slip conditions:

The next theorem comes from the papers [4] and [10] by J.N., P. Penel and Z. Skalák, P. Kučera, respectively:

Theorem 2. *Let Ω be a bounded or exterior domain in $\mathbb{R}^3$ with the boundary at least of the class $C^{2+(h)}$ for some $h>0$ or a half-space in $\mathbb{R}^3$. Let Ω' be a sub-domain of Ω, $-\infty < T_1 < T_2 < \infty$ and let $\mathbf{u}$ be a weak solution to the Navier-Stokes system (with $\mathbf{f}=\mathbf{0}$) in $\Omega \times (0,T)$, satisfying the no-slip boundary condition $\mathbf{u}=\mathbf{0}$ on $\partial\Omega \times (0,T)$ and condition (i) in $\Omega' \times (T_1,T_2)$. Let p be an associated pressure. Then $\partial_t\mathbf{u}$, ∇p and all their spatial derivatives (of all orders) are in $L^s(T_1+\delta, T_2-\delta;\, \mathbf{L}^\infty(\Omega''))$ for any $s \in (1,2)$, $T_1+\delta < T_2-\delta$ and $\Omega'' \subset\subset \Omega'$.*

The Navier–Stokes equations, the case $\Omega \neq \mathbb{R}^3$, Navier's boundary conditions:

One can take $s \in (0, 4]$ in Theorem 2. (See the paper [**??**] by P. Kučera, J.N., Z. Skalák.)

The Navier–Stokes equations, the case $\Omega \neq \mathbb{R}^3$, Navier–type boundary conditions:

One can take $s \in (0, \infty]$ in Theorem 2. (See the paper [5] by J.N., H. AlBaba.)

The MHD equations, either $\Omega = \mathbb{R}^3$ or $\Omega \neq \mathbb{R}^3$ and three types of boundary conditions for $\mathbf{u}$ + the Navier–type boundary conditions for $\mathbf{b}$:

The same results as for the Navier–Stokes equations, see the paper [6] by J.N. and M. Yang.

9. On some regularity criteria for weak solutions to the Navier –Stokes and MHD equations, based on the pressure

Put $\pi := p + \frac{1}{2}|\mathbf{b}|^2$.

The next theorem follows from Theorem 4 in [6]:

Theorem 3. *Let Ω be the same as in Theorem 1 or Theorem 2. Let $(\mathbf{u}, \mathbf{b}, p)$ be a suitable weak solution to the MHD system with $\mathbf{f} = \mathbf{g} = \mathbf{0}$, where $\mathcal{R}_f = \mathcal{R}_m$, with the no–slip boundary condition for $\mathbf{u}$ and the Navier–type boundary conditions for $\mathbf{b}$. Let Ω_1 be a sub-domain of Ω and $0 \le t_1 < t_2 \le T$. Let at least one of the following conditions hold:*

(a) $\pi \in L^\alpha(t_1, t_2;\, L^\beta(\Omega_1))$ *for some* $\alpha \in [1, \infty)$, $\beta \in (\frac{3}{2}, \infty]$, $2/\alpha + 3/\beta = 2$,

(b) $\nabla\pi \in L^\alpha(t_1, t_2;\, \mathbf{L}^\beta(\Omega_1))$ *for some* $\alpha \in [1, \infty)$, $\beta \in (1, \infty]$, $2/\alpha + 3/\beta = 3$.

Let $\Omega_2 \subset\subset \Omega_1$ and $0 < \delta < \frac{1}{2}(t_2 - t_1)$.
Then the solution $(\mathbf{u}, \mathbf{b}, \pi)$ is regular in $\Omega_2 \times (t_1 + \delta, t_2 - \delta)$.

If domain Ω is convex (and not such as in Theorem 1 or 2) then the statement on regularity can be extended to the whole domain Ω, up to the boundary. The corresponding result is the content of the next theorem, which follows from Theorem 5 in [**??**].

Theorem 4. *Let Ω be a convex domain in $\mathbb{R}^3$. Let $(\mathbf{u}, \mathbf{b}, p)$ be a suitable weak solution to the MHD system in $\Omega \times (T_1, T_2)$ with the no–slip boundary condition for $\mathbf{u}$ and the Navier–type boundary conditions for $\mathbf{b}$, where $\mathcal{R}_f = \mathcal{R}_m$. Let at least one of the conditions (a), (b) from Theorem 3 hold with $\Omega_1 = \Omega$ and $(T_1, T_2) = (0, T)$. Then the solution $(\mathbf{u}, \mathbf{b}, \pi)$ does not blow–up at any time instant $t \in (0, T]$.*

We assume that both $\mathbf{u}$ and $\mathbf{b}$ satisfy the Navier–type boundary conditions on $\partial\Omega \times (0, T)$ in the next criterion.

Recall that a function Φ on $[0, \infty)$ is said to be a *Young function* if it can be expressed in the form $\Phi(s) = \int_0^s \varphi(\sigma)\, \mathrm{d}\sigma$ for $s \geq 0$, where $\varphi(0) = 0$, $\varphi(\sigma) > 0$ for $\sigma > 0$, φ is right-continuous and non-decreasing on $[0, \infty)$ and $\lim_{\sigma \to \infty} \varphi(\sigma) = \infty$. (See e.g. [**?**].) It follows from this definition that the Young function Φ is continuous, non-negative, strictly increasing and convex on $[0, \infty)$, and

1) $\Phi(0) = 0,\ \lim_{s\to\infty} \Phi(s) = \infty$,

2) $\lim_{s\to 0+} \Phi(s)/s = 0,\ \lim_{s\to\infty} \Phi(s)/s = \infty$,

3) if $0 \le \alpha \le 1$ then $\Phi(\alpha s) \le \alpha\, \Phi(s)$ for all $s \ge 0$,

4) if $\beta > 1$ then $\Phi(\beta s) \ge \beta\, \Phi(s)$ for all $s \ge 0$,

see Lemma 4.2.2 in [8].

If D is a domain in $\mathbb{R}^3$ then the *Orlicz space* $L^\Phi(D)$ is the space of all measurable functions f on D with the finite norm, defined by

$$\|f\|_{L^\Phi(D)} \ := \ \inf\Big\{\lambda > 0;\ \int_D \Phi\Big(\frac{|f(\mathbf{x})|}{\lambda}\Big)\, \mathrm{d}\mathbf{x} \le 1\Big\}.$$

(This is the so called *Luxemburg norm* in $L^\Phi(D)$, see [8].)
We consider a Young function Φ that has these additional properties:

(ii) $s^{-3/2}\, \Phi(s)$ *is monotone increasing on* $[0, \infty)$ *and and tends to infinity as* $s \to \infty$,

(iii) $\Phi(s)^{-2/3} \in L^1((1, \infty))$.

The next theorem comes from the papers [1] and [7].

Theorem 5. *Suppose that*

- Ω *is a domain in* $\mathbb{R}^3$*, same as in Theorem 1 or Theorem 2,*
- *the Young function* Φ *satisfies the conditions (ii) and (iii),*
- $(\mathbf{u}, \mathbf{b}, p)$ *is a suitable weak solution of the MHD equations (with* $\mathbf{f} = \mathbf{g} = \mathbf{0}$*) in* $\Omega \times (0, T)$ *with the Navier–type boundary conditions for both the velocity and the magnetic field (in the case* $\Omega \neq \mathbb{R}^3$*),*
- Ω' *is a sub-domain of* Ω *and* $0 \leq T_1 < T_2 \leq T$*.*

Then the vector fields $\mathbf{u}$ *and* $\mathbf{b}$ *are Hölder continuous in* $\Omega' \times (T_1, T_2)$ *if at least one of the following conditions holds;*

1) $p_- \in L^\infty(T_1, T_2;\, L^\Phi(\Omega'))$,

2) $\mathcal{B}_+ \in L^\infty(T_1, T_2;\, L^\Phi(\Omega'))$*, where* $\mathcal{B} := p + \frac{1}{2}|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{b}|^2$.

The subscripts "$-$" and "$+$" denote the negative and non-negative parts, respectively.

References

1. U. J. Choe, J. Neustupa, M. Yang: Improved regularity criteria for the MHD equations in terms of pressure using an Orlicz norm. *Applied Mathematics Letters* **132**, October 2022, Paper No. 108121.
2. P. Kučera, J. Neustupa, Z. Skalák: The interior regularity of pressure associated with a weak solution to the Navier-Stokes equations with Navier's boundary conditions.
3. A. Mahalov, B. Nicolaenko, T. Shilkin: $L_{3,\infty}$ solutions to the MHD equations. *J. of Math. Sci.* **143** (2007), 2911—2923.
4. J. Neustupa, P. Penel: Anisotropic and geometric criteria for interior regularity of weak solutions to the 3D Navier-Stokes equations. In *Mathematical Fluid Mechanics, Recent Results and Open Questions,* ed. J. Neustupa and P. Penel, Birkhäuser, Basel 2001, pp. 237–268.
5. J. Neustupa, H. Al Baba: The interior regularity of pressure associated with a weak solution to the Navier-Stokes equations with the Navier-type boundary conditions. *J. Math. Anal. Appl.* **463** (2018), no. 1, 222–234.
6. J. Neustupa, M. Yang: On the pressure in the theory of MHD Equations. *Nonlinear Analysis: Real World Applications* **60** (2021), Article No. 103283.
7. J. Neustupa, M. Yang: New regularity criteria for weak solutions to the MHD equations in terms of an associated pressure. *J. Math. Fluid Mech.* **23** (2021), Article No. 73, https://doi.org/10.1007/s00021-021-00597-9.
8. L. Pick, A. Kufner, O. John, S. Fučík: *Function Spaces I.*

9. J. Serrin: On the interior regularity of weak solutions of the Navier-Stokes equations. *Arch. Rat. Mech. Anal.* **9** (1962) 187–195.
10. Z. Skalák, P. Kučera: Regularity of pressure in the neighbourhood of regular points of weak solutions of the Navier-Stokes equations. *Appl. Math.* **48** (2003), no. 6, 573–586.