# Brownian Motions, Itô Calculus, and SDE

Hyunchul Park, SUNY New Paltz

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# Warming Up - What Can One Prove in Analysis using Probability?

# Chapter 1

# Stone-Weierstrass Theorem

Probability is one branch of mathematics that studies *randomness*. It has close connections to other areas of mathematics and is very useful to solve problems in different areas of mathematics. As an illustration, we prove the Stone-Weierstrass theorem using binomial distributions and Chebyshev inequality.
Here is a statement of the theorem.

**Theorem 1.0.1.** *Let $f(x)$ be continuous on $[0,1]$. Then, for any $\epsilon > 0$ there exists a polynomial $B_n(x)$ such that*

$$\sup_{x\in[0,1]} |f(x) - B_n(x)| < \epsilon.$$

**Proof.** Let $X_i$ be iid Bernoulli random variables with $\mathbb{P}(X_i = 1) = p \in (0,1)$ and $S_n = X_1 + \cdots X_n$. Note that the distribution of $S_n$ is a binomial distribution with parameters $n$ and $p$. We define

$$B_n(p) := \mathbb{E}[f(\frac{S_n}{n})] = \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k} p^k(1-p)^{n-k}.$$

Since $f(x)$ is uniformly continuous on $[0,1]$, for given $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ if } |x - y| < \delta.$$

Let $K = \sup_{x\in[0,1]} |f(x)| < \infty$. Then, we have

$$\begin{aligned}
|B_n(p) - f(p)| &= \left|\mathbb{E}[f(\frac{S_n}{n})] - f(p)\right| \\
\leq& \mathbb{E}[\left|f(\frac{S_n}{n}) - f(p)\right|] \\
\leq& \mathbb{E}[\left|f(\frac{S_n}{n}) - f(p)\right|, |\frac{S_n}{n} - p| < \delta] + \mathbb{E}[\left|f(\frac{S_n}{n}) - f(p)\right|, |\frac{S_n}{n} - p| \geq \delta].
\end{aligned}$$

Note that

$$\mathbb{E}[\left|f(\frac{S_n}{n}) - f(p)\right|, |\frac{S_n}{n} - p| < \delta] \leq \epsilon \mathbb{P}(|\frac{S_n}{n} - p| < \delta) \leq \epsilon.$$

By the Chebyshev inequality and the fact that $\mathsf{Var}(S_n) = \dfrac{n\mathsf{Var}(X_1)}{n^2} = \dfrac{p(1-p)}{n} \leq \dfrac{1}{4n}$, we have

$$\mathbb{E}[\left| f(\frac{S_n}{n}) - f(p)\right|, |\frac{S_n}{n} - p| \geq \delta] \leq 2K\mathbb{P}(|\frac{S_n}{n} - p| \geq \delta) \leq \frac{2K}{\delta^2}\mathsf{Var}(\frac{S_n}{n}) \leq \frac{K}{2n\delta^2}.$$

Now take $n$ large so that $\dfrac{K}{2n\delta^2} < \epsilon$ and we obtain

$$\sup_{p\in[0,1]} |B_n(p) - f(p)| < 2\epsilon.$$

$\square$

# Part III

# Construction of Brownian Motions

# Chapter 2

# $L^1$-method; Broken Line Approximation

A *Brownian motion* or *Wiener process* is a stochastic process that satisfies the following conditions:

(1) $B_0 = 0$ a.s.;

(2) the increments $B_t - B_s$ have $N(0, t-s)$ distribution for all $0 \leq s \leq t$;

(3) the increments $B_{t_2} - B_{t_1}$ and $B_{t_4} - B_{t_3}$ are independent whenever $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$;

(4) the trajectories $t \to B_t$ are a.s. continuous.

The first question we must answer is *if there exist such processes?*

## 2.1 Observation on the Sample Paths of Brownian Motions

We start with a simple observation on the sample path of Brownian motions. Let $B = \{B_t\}_{t \geq 0}$ be a Brownian motions. Then, for each $t > 0$ we have

$$\begin{cases} B_t = B_{t/2} + (B_t - B_{t/2}) = B_{t/2} + \tilde{B}_{t/2}, \\ B_{t/2} = \dfrac{1}{2} B_t + \dfrac{1}{2}(B_{t/2} - \tilde{B}_{t/2}), \end{cases}$$

where $\tilde{B}_t = B_t - B_{t/2}$. Hence, $B_t$ is a sum of independent normal random variables with variance $\dfrac{t}{2}$, and $B_{t/2}$ is a sum of $\dfrac{1}{2} B_t$ and $\dfrac{1}{2}(B_{t/2} - \tilde{B}_{t/2})$, which is independent of $B_{t/2} + \tilde{B}_{t/2}$ as one can see from the following simple fact.

**Lemma 2.1.1.** *Let $X$ and $Y$ are independent normal distributions with parameters 0 and $\sigma^2$. Then, $X + Y$ and $X - Y$ are independent $N(0, 2\sigma^2)$ distributions.*

**Proof.** Note that

$$\mathbb{E}[e^{i\xi X}] = \mathbb{E}[e^{i\xi Y}] = e^{-\frac{\sigma^2|\xi|^2}{2}}.$$

Hence, we have

$$\mathbb{E}[e^{i\xi_1(X+Y)}e^{i\xi_2(X-Y)}] = \mathbb{E}[e^{i(\xi_1+\xi_2)X}e^{i(\xi_1-\xi_2)Y}] = e^{-\frac{\sigma^2(\xi_1+\xi_2)^2}{2}}e^{-\frac{\sigma^2(\xi_1-\xi_2)^2}{2}} = e^{-\sigma^2|\xi_1|^2}e^{-\sigma^2|\xi_2|^2}.$$

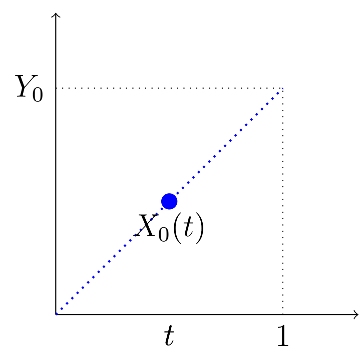
□

## 2.2 Reconstruction of Sample Paths using Broken-Line Approximation

Now let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with independent normal distributions $Y_0$ and $Y_n^k$, $n \in \mathbb{N}$, $k \in \{1, 2, \cdots, 2^{n-1}\}$ with $\mathrm{Var}(Y_n^k) = \dfrac{1}{2^{n+1}}$

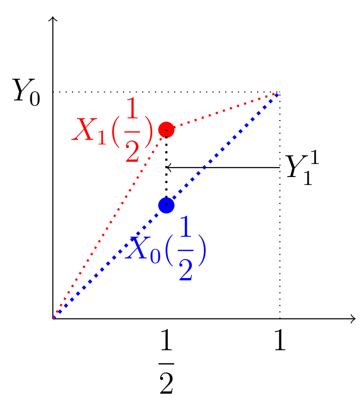
(1) We define $X_0(t)$, $t \in [0,1]$ as

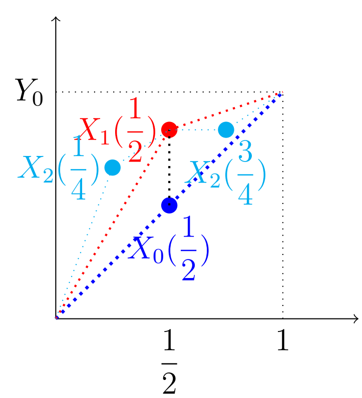
$$X_0(t) = tY_0.$$




(2) We define $X_1(t)$ as

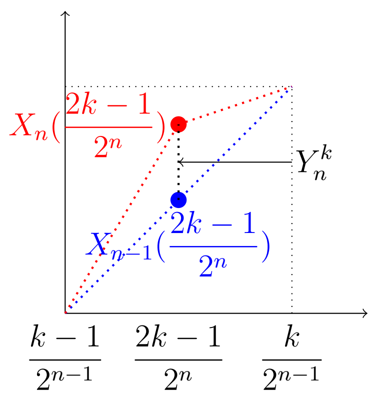
$$\begin{cases} X_1(0) = 0, \\ X_1(1) = X_0(1) = Y_0, \\ X_1(\frac{1}{2}) = \frac{1}{2}Y_0 + Y_1^1 = X_0(\frac{1}{2}) + Y_1^1, \\ X_1(t) \text{ is linear between these points.} \end{cases}$$



(3) In general, we repeat the process as follows:

$$\begin{cases} X_n(0) = 0, \\ X_n(1) = Y_0, \\ X_n(\frac{2k-1}{2^n}) = X_{n-1}(\frac{2k-1}{2^n}) + Y_n^k, \\ X_n(t) \text{ is linear between these points.} \end{cases}$$




(4) Note that by the construction we have

$$X_n(\frac{2k-1}{2^n}) - X_n(\frac{2k-2}{2^n}) = \frac{1}{2}\left(X_{n-1}(\frac{k}{2^{n-1}}) - X_{n-1}(\frac{k-1}{2^{n-1}})\right) + Y_n^k,$$

and

$$X_n(\frac{k}{2^{n-1}}) - X_n(\frac{2k-1}{2^n}) = \frac{1}{2}\left(X_{n-1}(\frac{k}{2^{n-1}}) - X_{n-1}(\frac{k-1}{2^{n-1}})\right) - Y_n^k.$$

Note that the variance of the two expressions above are both $\frac{1}{4}\frac{1}{2^{n-1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}$ and the two are independent of each other from Lemma 2.1.1

(5) Note that

$$\sup_{t\in[0,1]} |X_n(t) - X_{n-1}(t)| \leq \max_{1\leq k\leq 2^{n-1}} |X_n(\frac{k}{2^n}) - X_{n-1}(\frac{k}{2^n})|$$
$$= \max_{1\leq k\leq 2^{n-1}} |Y_n^k| \leq \left(\sum_{k=1}^{2^{n-1}} |Y_n^k|^4\right)^{1/4},$$

and by Jensen's inequality we have

$$\mathbb{E}[\sup_{t\in[0,1]} |X_n(t) - X_{n-1}(t)|] \leq \mathbb{E}\left[\left(\sum_{k=1}^{2^{n-1}} |Y_n^k|^4\right)^{1/4}\right] \leq \mathbb{E}\left[\sum_{k=1}^{2^{n-1}} |Y_n^k|^4\right]^{1/4}$$
$$\leq (2^{n-1}c(\frac{1}{2^{n+1}})^2)^{1/4} = c^{1/4}2^{-\frac{n+3}{4}},$$

where we used the fact that $\mathbb{E}[N^{2n}] = c_n\sigma^{2n}$ for $N \sim N(0,\sigma^2)$ and some constant $c_n$. This implies that

$$\sum_{n=1}^{\infty} \mathbb{E}[\sup_{t\in[0,1]} |X_n(t) - X_{n-1}(t)|] < \infty.$$

This shows the following.

**Theorem 2.2.1.** *The processes $X_n(t)$ converge a.s. and in $L^1$ to a continuous process $X(t)$ so that*

$$\mathbb{E}[\sup_{t\in[0,1]} |X_n(t) - X(t)|] \to 0, \quad n \to \infty,$$

*and also a.s.*

$$\sup_{t\in[0,1]} |X_n(t) - X(t)| \to 0, \quad n \to \infty,$$

*The limiting process $X(t)$ is a standard Brownian Motion on $[0,1]$.*

## 2.3 Exercises

(1) Let $X$ be a standard normal distribution.

 (a) Show that the moment generating function is $M_X(\lambda) = \mathbb{E}[e^{\lambda X}] = e^{\frac{\lambda^2}{2}}$ for all $\lambda \in \mathbb{R}$.
 (b) Show that

$$\mathbb{E}[X^{2n+1}] = 0, \quad \mathbb{E}[X^{2n}] = \frac{(2n)!}{2^n n!}, n \in \mathbb{N}.$$

(2) Prove Theorem 2.2.1 as follows:

 (a) Show that $\{X_n(t)\}$ is uniformly Cauchy in $L^1(\Omega,\mathbb{P})$, that is, for any $\epsilon > 0$ there exists $N = N(\epsilon)$ such that

$$\mathbb{E}[|X_n(t) - X_m(t)|] < \epsilon \text{ for all } n,m \geq N \text{ and } t \in [0,1].$$

 Hence, $X_t = \lim_{n\to\infty} X_n(t)$ exists in $L^1(\Omega,\mathbb{P})$ and $\lim_{n\to\infty} \mathbb{E}[\sup_{t\in[0,1]} |X_n(t) - X(t)|] = 0$.

 (b) Let $N$ be a standard normal random variable. Then, for any constant $A > 0$ we have

$$\mathbb{P}(|N| \geq A) \leq e^{-\frac{A^2}{2}},$$

 *Solution.*

$$\begin{aligned}
\mathbb{P}(N \geq A) &= \int_A^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+A)^2}{2}} dx \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times e^{-\frac{2xA+A^2}{2}} dx \\
&\leq \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times e^{-\frac{A^2}{2}} dx = \frac{1}{2} \times e^{-\frac{A^2}{2}}.
\end{aligned}$$

□

(c) Use Borel-Cantelli Lemma to show that for a.e. $\omega \in \Omega$ there exists $N = N(\omega)$ such that

$$|Y_n^k| \le \sqrt{n} 2^{-\frac{n+1}{2}} \text{ for all } n \ge N(\omega).$$

*Solution.* Note that

$$\mathbb{P}(2^{\frac{n+1}{2}}|Y_n^k| \ge \sqrt{n}) \le e^{-\frac{n}{2}} \text{ for all } n \ge 1,$$

and by Borel-Cantelli Lemma the result follows. □

(d) Show that $X_n(t)$ converges uniformly to $X_t$ a.s. (Hence, $t \to X_t$ is continuous a.s.).

*Solution.* From the previous question, for a.e. $\omega$ there exists $N(\omega)$ such that for all $n \ge N$

$$\sup_{t\in[0,1]} |X_n(t) - X_{n-1}(t)| \le \max_{1\le k\le 2^{n-1}} |X_n(\frac{k}{2^n}) - X_{n-1}(\frac{k}{2^n})| \le= \max_{1\le k\le 2^{n-1}} |Y_n^k| \le \sqrt{n} 2^{-\frac{n+1}{2}}.$$

□

(e) Show that $X_{t+s} - X_t$ is independent of $\sigma(X_u, u \le t)$ and has $N(0, s)$ distribution.

*Solution.* It is enough to show that

$$\mathbb{E}[e^{i\xi_1(X_{t+s}-X_t)} e^{i\xi_2 X_u}] = e^{-\frac{\xi_1^2}{2}s} e^{-\frac{\xi_2^2}{2}u} \text{ for any } u \le t \text{ and } \xi_1, \xi_2 \in \mathbb{R}.$$

This is true if all $t, s, u$ are dyadic integers. For a general case, take sequences of dyadic integers $\{t_n\}, \{s_n\}$ and $\{u_n\}$ such that $\lim t_n = t, \lim s_n = s, \lim u_n = u$. By continuity, $\lim X_{t_n} = X_t$, $\lim X_{s_n} = X_s$, and $\lim X_{u_n} = X_u$ and it follows from the dominated convergence theorem

$$\mathbb{E}[e^{i\xi_1(X_{t+s}-X_t)} e^{i\xi_2 X_u}] = \lim_{n\to\infty} \mathbb{E}[e^{i\xi_1(X_{t_n+s_n}-X_{t_n})} e^{i\xi_2 X_{u_n}}]$$
$$= \lim_{n\to\infty} e^{-\frac{\xi_1^2}{2}s_n} e^{-\frac{\xi_2^2}{2}u_n} = e^{-\frac{\xi_1^2}{2}s} e^{-\frac{\xi_2^2}{2}u}.$$

□

# Chapter 3

# $L^2$-method; Gaussian White Noise

Now we turn our attention to constructing Brownian motions. We will construct the Gaussian white noise, which is an isometry from $L^2([0,1], dx)$ into a space of centered Gaussian distributions. Brownian motions $W = \{W_t\}_{t\geq 0}$ will be constructed as an image of $1_{[0,t]}(x)$ under this Gaussian white noise.

For every $t \in [0,1]$, we define *Haar functions*

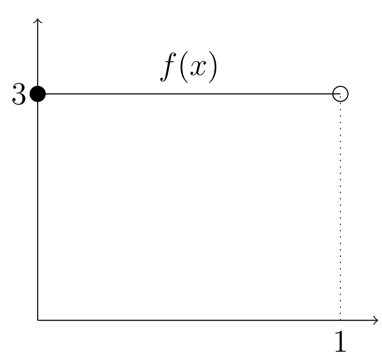
$$h_0(t) = 1, \quad t \in [0,1],$$

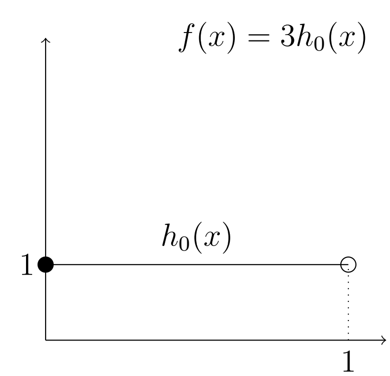
and

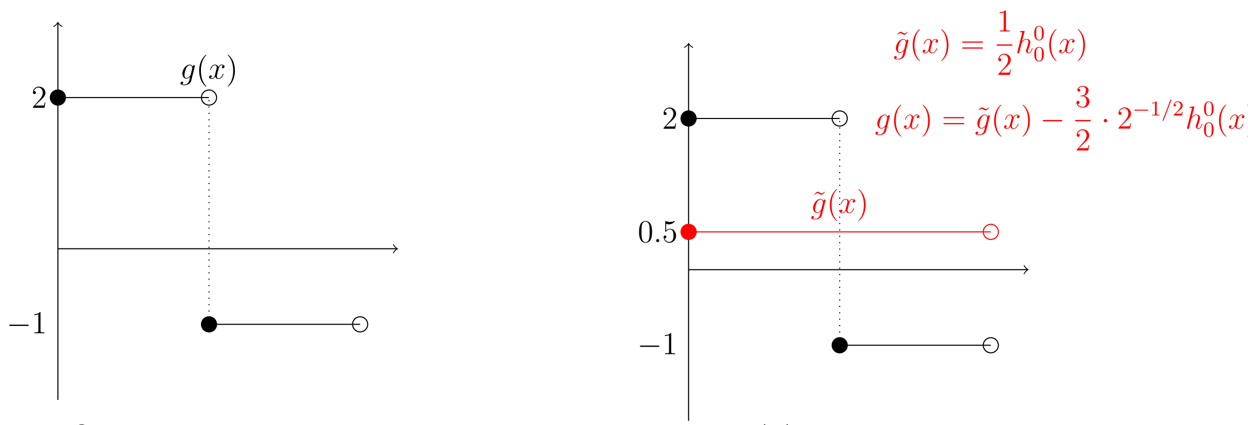
$$h_k^n(t) = 2^{n/2} \times 1_{\{[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}})\}}(t) - 2^{n/2} \times 1_{\{[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}})\}}(t),$$

where $n \in \{0,1,2,\cdots\}$ and $k \in \{0,1,2,\cdots,2^n-1\}$.

(1) Show that $\{h_0, h_k^n\}$, $n \in \{0,1,2,\cdots\}$ and $k \in \{0,1,2,\cdots,2^n-1\}$ form an orthonormal basis for $L^2([0,1], \mathcal{B}[0,1], dt)$.






**Proof.** It is enough to show that for any step function $f(x)$ which has constant values on intervals of the form $[\frac{k-1}{2^n}, \frac{k}{2^n})$, $k \in \{1, 2, \cdots, 2^n\}$ can be represented as a linear combination of $h_0$ and $h_k^j$, $j \in \{0, 1, 2, \cdots, n-1\}$ since polynomials can be approximated by these step functions and by Stone-Weierstrass Theorem (Theorem 1.0.1) polynomials are dense in the space of continuous functions for uniform norm on $[0, 1]$.

We prove this using a mathematical induction. This is trivial when $n = 0$. Suppose this holds for some $n-1$. Let $f(x)$ be a step function whose values are constants on intervals of the form $[\frac{k-1}{2^{n+1}}, \frac{k}{2^{n+1}})$, $k \in \{1, 2, \cdots, 2^{n+1}\}$. Define an ancestor $\tilde{f}(x)$ of $f(x)$ whose values are constant on intervals of the form $[\frac{k-1}{2^n}\frac{k}{2^n})$ and the values are determined by the average of $f(x)$. That is,

$$\tilde{f}(x) = \frac{f(\frac{2k-2}{2^{n+1}}) + f(\frac{2k-1}{2^{n+1}})}{2}, \quad x \in [\frac{k-1}{2^n}\frac{k}{2^n}).$$

By the induction hypothesis $\tilde{f}(x)$ can be represented as a linear combination of $h_0$ and $h_k^j$, $j \in \{0, 1, 2, \cdots, n-1\}$. Then, for each interval of the form $[\frac{2l-2}{2^{n+1}}, \frac{2l}{2^{n+1}})$, let

$$\begin{aligned} & g_{l-1}^n(x) \\ = & \left(f(\frac{2l-2}{2^{n+1}})) - \tilde{f}(\frac{l-1}{2^n})\right)\frac{h_{l-1}^n(x)}{2^{n/2}} \\ = & \frac{1}{2}\left(f(\frac{2l-2}{2^{n+1}})) - f(\frac{2l-1}{2^n})\right)\frac{h_{l-1}^n(x)}{2^{n/2}}. \end{aligned}$$

Finally define

$$\begin{aligned} g(x) &= \tilde{f}(x) + \sum_{l=1}^{2^n} g_{l-1}^n(x) \\ &= \tilde{f}(x) + \sum_{l=1}^{2^n} \frac{1}{2}\left(f(\frac{2l-2}{2^{n+1}})) - f(\frac{2l-1}{2^n})\right)\frac{h_{l-1}^n(x)}{2^{n/2}}. \end{aligned}$$

Then it is easy to observe that

$$\begin{aligned}
& g(\frac{2l-2}{2^{n+1}}) \\
= \;& \tilde{f}(\frac{l-1}{2^n}) + \frac{1}{2}\left(f(\frac{2l-2}{2^{n+1}})) - f(\frac{2l-1}{2^n})\right) \\
= \;& \frac{1}{2}\left(f(\frac{2l-2}{2^{n+1}}) + f(\frac{2l-1}{2^{n+1}}\right) + \frac{1}{2}\left(f(\frac{2l-2}{2^{n+1}})) - f(\frac{2l-1}{2^n})\right) \\
= \;& f(\frac{2l-2}{2^{n+1}}),
\end{aligned}$$

and

$$\begin{aligned}
& g(\frac{2l-1}{2^{n+1}}) \\
= \;& \tilde{f}(\frac{l-1}{2^n}) + \frac{1}{2}\left(f(\frac{2l-2}{2^{n+1}})) - f(\frac{2l-1}{2^n})\right) \times (-1) \\
= \;& \frac{1}{2}\left(f(\frac{2l-2}{2^{n+1}}) + f(\frac{2l-1}{2^{n+1}}\right) - \frac{1}{2}\left(f(\frac{2l-2}{2^{n+1}})) - f(\frac{2l-1}{2^n})\right) \\
= \;& f(\frac{2l-1}{2^{n+1}}).
\end{aligned}$$

Hence $g(x) = f(x)$ for all $x \in [0,1]$. □

Let $(E, \mathcal{E})$ be a measurable space, and let $\mu$ be a $\sigma$-finite measure on $(E, \mathcal{E})$. A *Gaussian white noise* with intensity $\mu$ is an isometry $G$ from $L^2(E, \mathcal{E}, \mu)$ into a centered Gaussian space.

(2)

Suppose that $\mathcal{N}_0$, $\mathcal{N}_k^n$, $n \in \{0, 1, 2, \cdots\}$ and $k \in \{0, 1, 2, \cdots, 2^n - 1\}$, are independent, standard random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that there exists a Gaussian white noise such that

$$G(h_0) = \mathcal{N}_n, \quad \text{and } G(h_k^n) = \mathcal{N}_k^n.$$

**Proof.** For each $f \in L^2([0,1], dx)$ can be written uniquely as

$$f(x) = c_0 h_0 + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} c_k^n h_k^n$$

with

$$\|f\|_2^2 = c_0^2 + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} (c_k^n)^2 < \infty.$$

Let $G(f_m) = c_0\mathcal{N}_0 + \sum_{n=0}^{m}\sum_{k=0}^{2^n-1} c_k^n \mathcal{N}_k^n$. Note that by independence of $\mathcal{N}$ and $\mathcal{N}_k^n$

$$\mathbb{E}[(G(f_m) - G(f_l))^2] = \mathbb{E}\left[\sum_{n=m+1}^{l}\sum_{k=0}^{2^n-1} (c_k^n)^2(\mathcal{N}_k^n)^2\right] = \sum_{n=m+1}^{l}\sum_{k=0}^{2^n-1} (c_k^n)^2 \to 0$$

as $m, l \to \infty$. Hence $G(f_m)$ is Cauchy in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. We denote its limit by

$$G(f) = \lim_m G(f_m) = c_0\mathcal{N}_0 + \sum_{n=0}^{\infty}\sum_{k=0}^{2^n-1} c_k^n \mathcal{N}_k^n.$$

Clearly this $G$ has the desired property. $\square$

(3) For each $t \in [0,1]$ set $B_t := G(1_{[0,t]})$. Show that

$$B_t = t\mathcal{N}_0 + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{2^n-1} g_k^n(t)\mathcal{N}_k^n\right),$$

where

$$g_k^n(t) = \int_0^t h_k^n(s)ds.$$

$g_k^n(t)$ are called *Schauder functions*.

**Proof.** Write

$$1_{[0,t]} = c_0 h_0 + \sum_{n=0}^{\infty}\sum_{k=0}^{2^n-1} c_k^n h_k^n.$$

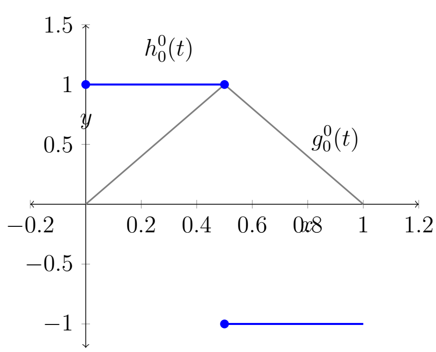
Then

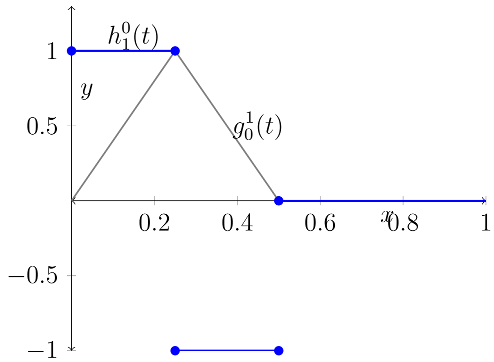
$$c_0 = \langle 1_{[0,t]}, h_0\rangle = \int_0^1 1_{[0,t]}(s)ds = t,$$

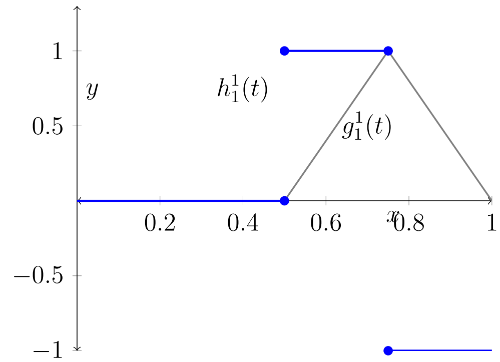
and

$$c_k^n = \langle 1_{[0,t]}, h_k^n\rangle = \int_0^t h_k^n(s)ds.$$

$\square$






(4) In this step, we show that $B_t^m$ converges uniformly to $B_t$. A key ingredient is the following *Borel-Cantelli lemma*.

> **Lemma 3.0.1** (Borel-Cantelli lemma)**.** *Let $A_n$ be events on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$. Then, $\mathbb{P}(\limsup_n A_n) = 0$.*

**Proof.** From $\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$, we have $\limsup_n A_n \subset \bigcup_{m=n}^{\infty} A_m$ for any $n \in \mathbb{N}$. Now the conclusion follows immediately from

$$\mathbb{P}(\limsup_n A_n) \leq \mathbb{P}(\bigcup_{m=n}^{\infty} A_m) \leq \sum_{m=n}^{\infty} \mathbb{P}(A_m) \to 0.$$

□

> For each $m \geq 0$ and $t \in [0, 1]$ define
>
> $$B_t^m = t\mathcal{N}_0 + \sum_{n=0}^{m} \left( \sum_{k=0}^{2^n - 1} g_k^n(t) \mathcal{N}_k^n \right).$$
>
> Show that $B_t^m$ converges uniformly to $B_t$ on $[0, 1]$ almost surely.

**Proof.** The key idea is to choose a clever choice of $c_n$ with $c_n \to 0$ so that

$$\sum_n \mathbb{P} \left( \sum_{k=0}^{2^n - 1} g_k^n(t) \mathcal{N}_k^n > c_n \right) < \infty,$$

and use Borel-Cantelli lemma.

First note that for the standard normal random variable $\mathcal{N}$ and $c \geq 1$ we have

$$\mathbb{P}(|\mathcal{N}| \geq c) = 2 \int_c^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \leq \sqrt{\frac{2}{\pi}} \int_c^{\infty} x e^{-\frac{x^2}{2}} dx = \sqrt{\frac{2}{\pi}} e^{-\frac{c^2}{2}}.$$

Note that supports of functions $g_k^n(t)$ are all disjoint and $|g_k^n(t)| \leq \frac{2^{n/2}}{2^n} = 2^{-\frac{n}{2}}$. Hence we have

$$\begin{aligned} &\mathbb{P}\left(\sum_{k=0}^{2^n-1} g_k^n(t)\mathcal{N}_k^n > c_n\right) \leq \mathbb{P}(2^{-\frac{n}{2}} \sup_{0\leq k\leq 2^n-1} \mathcal{N}_k^n > c_n) \\ = \ &\mathbb{P}\left(\sup_{0\leq k\leq 2^n-1} \mathcal{N}_k^n > 2^{\frac{n}{2}}c_n\right) \leq \sqrt{\frac{2}{\pi}}\exp(-\frac{c_n^2 2^n}{2}). \end{aligned}$$

Now let $c_n = 2^{-\frac{n}{4}}$ so that $c_n \to 0$ and $\sum_n \exp(-\frac{c_n^2 2^n}{2}) = \sum_n \exp(-\frac{2^{n/2}}{2}) < \infty$.

Now it follows from Borel-Cantelli lemma we have

$$\mathbb{P}\left(\limsup\{\sum_{k=0}^{2^n-1} g_k^n(t)\mathcal{N}_k^n > 2^{-\frac{n}{4}}\}\right) = 0.$$

Hence for almost every $\omega \in \Omega$ there exists $N = N(\omega)$ such that

$$\sum_{k=0}^{2^n-1} g_k^n(t)\mathcal{N}_k^n \leq 2^{-\frac{n}{4}}$$

for all $n \geq N(\omega)$. This shows that $B_t^m$ converges uniformly for all $t \in [0,1]$. □

(5) Hence we can, for every $t \in [0,1]$, select a random variable $B_t'$ which is a.s. equal to $B_t$, in such a way that the mapping $t \mapsto B_t'(\omega)$ is continuous for every $\omega \in \Omega$.

**Proof.** Since the uniform limit of continuous functions is continuous, $B_t$ is continuous on $\Omega'$ with $\mathbb{P}(\Omega') = 1$. Define

$$B_t'(\omega) = \begin{cases} B_t(\omega) & \text{if } \omega \in \Omega', \\ 0 & \text{if } \omega \notin \Omega'. \end{cases}$$

□

# Chapter 4

# Probabilistic Solution to Dirichlet Problem

Let $D$ be a domain in $\mathbb{R}^d$. We say a point $z \in \partial D$ is a *regular boundary point* if $\mathbb{P}_z(\tau_D = 0) = 1$. Let $(\partial D)_r$ be a collection of all regular boundary points. A domain $D$ is called regular if $(\partial D)_r = \partial D$.

**Theorem 4.0.1.** *For any domain (bounded or unbounded) $D$ and any $f \in L^\infty(\partial D)$, the function $H_D f$ defined in $\mathbb{R}^d$ by*

$$H_D f(x) = \mathbb{E}_x[\tau_D < \infty, f(W_{\tau_D})]$$

*is harmonic in $D$. If, in addition, $z \in (\partial D)_r$ and $f$ is continuous at $z$, then*

$$\lim_{x \to z, x \in D} H_D f(x) = f(z).$$

Before proving the theorem we need to recall some facts. If follows from [1, Theorem 1.17] we have

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}_x[\tau_D] \leq A_d |D|^{2/d}, \quad A_d = \frac{d+2}{2\pi d}(\frac{d+2}{2})^{2/d}. \tag{4.1}$$

In particular, if $|D| < \infty$, then $\mathbb{E}_x[\tau_D] < \infty$ a.s.

**Lemma 4.0.2.** *Let $D \subset \mathbb{R}^d$ and $B$ is an open ball with $\bar{B} \subset D$. Then, we have*

1. $\tau_B + \tau_D \cdot \theta_{\tau_B} = \tau_D$.
2. $W_{\tau_D} \cdot \theta_{\tau_B} = W_{\tau_D}$.
3. *Let* $\Phi = 1_{\{\tau_D < \infty\}} f(W_{\tau_D})$. *Then,* $\Phi = \Phi \cdot \theta_{\tau_B}$.

**Proof.** First, it follows from (4.1) $\tau_B < \infty$ a.s. Note that for all such $\omega \in \Omega$ with $\tau_B(\omega) < \infty$

we have

$$\begin{aligned}
&\tau_D \cdot \theta_{\tau_B}(\omega) \\
=\;& \inf\{t>0 : X_t(\theta_{\tau_B}(\omega)) \notin D\} \\
=\;& \inf\{t>0 : \theta_{\tau_B}(\omega)(t) \notin D\} \\
=\;& \inf\{t>0 : \omega(\tau_B(\omega)+t) \notin D\} \\
=\;& \tau_D(\omega) - \tau_B(\omega).
\end{aligned}$$

Second, note that for any stopping time we define $X_\tau(\omega) := X(\tau(\omega), \omega)$. Hence, we have

$$\begin{aligned}
&W_{\tau_D} \cdot \theta_{\tau_B}(\omega) \\
=\;& X(\tau_D(\theta_{\tau_B}(\omega)), \theta_{\tau_B}(\omega)) \\
=\;& X(\tau_B(\omega) + \tau_D(\theta_{\tau_B}(\omega)), \omega) \\
=\;& X(\tau_D(\omega), \omega) \\
=\;& W_{\tau_D}(\omega),
\end{aligned}$$

where we used $\tau_B + \tau_D \cdot \theta_{\tau_B} = \tau_D$ in the middle.
Third, note that

$$\Phi \cdot \theta_{\tau_B}(\omega) = 1_{\{\tau_D<\infty\}}(\theta_{\tau_B}(\omega)) f(W_{\tau_D}(\theta_{\tau_B}(\omega)).$$

From the second, we have $W_{\tau_D}(\theta_{\tau_B}(\omega)) = W_{\tau_D}(\omega)$. Note that $1_{\{\tau_D<\infty\}}(\omega) = 1$ if and only if $\tau_D(\omega) < \infty$. Observe that $1_{\{\tau_D<\infty\}}(\theta_{\tau_B}(\omega)) = 1$ if and only if

$$\tau_D(\theta_{\tau_B}(\omega)) = \tau_D(\omega) - \tau_B(\omega) < \infty.$$

Since $\tau_B < \infty$ a.s., we conclude that $1_{\{\tau_D<\infty\}}(\theta_{\tau_B}(\omega)) = 1$ if and only if $\tau_D(\omega) < \infty$ a.s. $\square$

**Proof of Theorem 4.0.1** We first prove that $H_D f(x)$ is harmonic in $D$ by showing that it has a sphere averaging property. Let $x \in D$ and $B = B(x,r)$ with $\bar{B} \subset D$. By Lemma 4.0.2 and the strong Markov property at $\tau_B$ we have

$$\begin{aligned}
&H_D f(x) = \mathbb{E}_x[\Phi] = \mathbb{E}_x[\Phi \cdot \theta_{\tau_B}] \\
=\;& \mathbb{E}_x[\mathbb{E}_{X_{\tau_B}}[\Phi]] \\
=\;& \int_{S(x,r)} \mathbb{E}_u[\Phi] \mathbb{P}_x(X_{\tau_{B(x,r)}} \in du) \\
=\;& \frac{1}{\sigma(S(x,r))} \int_{S(x,r)} \mathbb{E}_u[\Phi] \sigma(du) \\
=\;& \frac{1}{\sigma(S(x,r))} \int_{S(x,r)} H_D f(u) \sigma(du),
\end{aligned}$$

where we used the fact that the distribution of $\mathbb{P}_x(X_{\tau_{B(x,r)}} \in du)$ is a uniform measure on $S(x,r)$.
Now fix $z \in (\partial D)_r = \{w \in \mathbb{R}^d : \mathbb{P}_w(\tau_D = 0) = 1\}$. Given $\epsilon > 0$ take $\delta_1 = \delta_1(\epsilon)$ such that $|f(x) - f(z)| < \epsilon$ for all $|x - z| < \delta_1$. Note that

$$\begin{aligned}
&\mathbb{E}_x[\tau_D < \infty, |f(W_{\tau_D}) - f(z)|] \\
=\;& \mathbb{E}_x[\tau_D < \infty, |f(W_{\tau_D}) - f(z)|, \tau_{B(z,\delta)} > \tau_D] + \mathbb{E}_x[\tau_D < \infty, |f(W_{\tau_D}) - f(z)|, \tau_{B(z,\delta)} \leq \tau_D] \\
\leq\;& \epsilon + \mathbb{E}_x[\tau_D < \infty, |f(W_{\tau_D}) - f(z)|, \tau_{B(z,\delta)} \leq \tau_D],
\end{aligned}$$

where we used the fact on $\{\tau_{B(z,\delta)} > \tau_D\}$ $W_{\tau_D} \in B(z,r)$ and $|f(W_{\tau_D}) - f(z)| < \epsilon$. The second expression above can be bounded above by

$$\mathbb{E}_x[\tau_D < \infty, |f(W_{\tau_D}) - f(z)|, \tau_{B(z,\delta)} \le \tau_D] \le 2\|f\|_\infty \mathbb{P}_x(\tau_{B(z,\delta)} \le \tau_D).$$

The intuitive idea is that when $x$ is near $z \in (\partial D)_r$, $\tau_D$ must be small and this makes the probability small. For $x \in B(z,\delta)$ we have $B(x,\delta/2) \subset B(z,\delta)$ and $\tau_{B(x,\delta/2)} \le \tau_{B(z,\delta)}$. Hence, we have

$$\mathbb{P}_x(\tau_{B(z,\delta)} \le \tau_D) \le \mathbb{P}_x(\tau_{B(x,\delta/2)} \le \tau_D).$$

By the path continuity we have $\mathbb{P}_x(\tau_{B(x,\delta/2)} > 0) = 1$ or $\mathbb{P}_x(\tau_{B(x,\delta/2)} = 0) = 0$. Since $\lim_{n\to\infty} \mathbb{P}_x(\tau_{B(x,\delta/2)} \le 1/n) = \mathbb{P}_x(\tau_{B(x,\delta/2)} = 0) = 0$, we can take $s > 0$ such that

$$\mathbb{P}_x(\tau_{B(x,\delta/2)} \le s) < \epsilon. \tag{4.2}$$

Fix this $s > 0$. Note that

$$\begin{aligned} & \mathbb{P}_x(\tau_{B(z,\delta)} \le \tau_D) \\ \le\ & \mathbb{P}_x(\tau_{B(x,\delta/2)} \le s \text{ or } \tau_D > s) \\ \le\ & \mathbb{P}_x(\tau_{B(x,\delta/2)} \le s) + \mathbb{P}_x(\tau_D > s). \end{aligned}$$

Since $z \in (\partial D)_r$ we have $\mathbb{P}_z(\tau_D > s) = 0$. The map $x \to \mathbb{P}_x(\tau_D > s)$ is upper-semi-continuous and we have

$$\limsup_{x\to z} \mathbb{P}_x(\tau_D > s) \le \mathbb{P}_z(\tau_D > s) = 0.$$

Hence, $\lim_{x\to z} \mathbb{P}_x(\tau_D > s) = 0$ or

$$\lim_{x\to z} \mathbb{P}_x(\tau_D \le s) = 1. \tag{4.3}$$

Take $\delta_2 = \delta_2(\epsilon)$ such that

$$\mathbb{P}_x(\tau_D > s) < \epsilon \tag{4.4}$$

for all $x \in \bar{D}$ with $|x - z| < \delta_2$.
Now let $\delta = \min(\delta_1, \delta_2)$. For any $x \in \bar{D}$ with $|x - z| < \delta$ we have from **(4.2)** and **(4.4)**

$$\begin{aligned} & \mathbb{E}_x[\tau_D < \infty, |f(W_{\tau_D}) - f(z)|] \\ \le\ & \epsilon + \mathbb{E}_x[\tau_D < \infty, |f(W_{\tau_D}) - f(z)|, \tau_{B(z,\delta)} \le \tau_D] \\ \le\ & \epsilon + 2\|f\|_\infty \mathbb{P}_x(\tau_{B(z,r)} \le \tau_D) \\ \le\ & \epsilon + 2\|f\|_\infty \left(\mathbb{P}_x(\tau_{B(x,\delta/2)} \le s) + \mathbb{P}_x(\tau_D > s)\right) \\ \le\ & \epsilon + 4\epsilon\|f\|_\infty. \end{aligned}$$

Hence, we have

$$\lim_{x\to z, x\in\bar{D}} H_D f(x) = \lim_{x\to z, x\in\bar{D}} \mathbb{P}_x(\tau_D < \infty) \cdot f(z).$$

Finally, it follows from **(4.3)** we have $\lim_{x\to z, x\in\bar{D}} \mathbb{P}_x(\tau_D < \infty) = 1$. □

It is well-known that for Lipschitz domain $D$, all boundary points are regular for Brownian motions. Hence, we have the following theorem.

**Theorem 4.0.3.** *Let $D \subset \mathbb{R}^d$ be a bounded Lipchitz domain and $f$ is continuous on $\partial D$. Then, there exists a unique solution to the following Dirichlet problem*

$$\begin{cases} \Delta u(x) = 0, & x \in D, \\ u(z) = f(z), & z \in \partial D. \end{cases}$$

*Furthermore, $u$ is given by*

$$u(x) = \mathbb{E}_x[f(W_{\tau_D})].$$

# Part IV

# Stochastic Integrals with respect to Brownian Motions

# Chapter 5

# Stochastic Integral with respect to Brownian Motions

In this chapter, we will define the stochastic integral (Itô integral) $\mathcal{I}(f) = \int_0^T f(t,\omega)dW_t$. Note that $\mathbb{E}[e^{i\theta W_t}] = e^{-t|\theta|^2}/2$ and this shows that $W_t$ and $t^{1/2}W_1$ has the same distribution. This means that locally $W_t$ moves as fast as $\sqrt{t}$ and $W_t$ cannot be of bounded variation as $\sum \frac{1}{\sqrt{n}} = \infty$. Hence, the Lebesgue-Stieltjes integral does not work as the sample $t \to W_t$ is not of bounded variation. We will overcome this by defining the stochastic integral as an element in $L^2(\mathbb{P})$.

There are a few steps to achieve this:

1. Define $\mathcal{I}(f)$ when $f$ is *elementary*.
2. Use the Itô's isometry $\mathbb{E}[\mathcal{I}(f)^2] = \mathbb{E}[\int_0^T f(t,\omega)^2 dt]$ to extend $\mathcal{I}(f)$ for $f \in L^2((0,T) \otimes \mathbb{P})$

**Definition 5.0.1.** *Fix $T > 0$. We define $V = V(T)$ be a collection of functions $f$ such that*

*(1) $(t,\omega) \to f(t,\omega)$ is $\mathcal{B} \otimes \mathcal{F}$-measurable.*

*(2) For each $t > 0$, $f(t,\omega) \in \mathcal{F}_t$ ($\mathcal{F}_t$-adapted).*

*(3) $\mathbb{E}[\int_0^T f(t,\omega)^2 dt] < \infty$.*

**Definition 5.0.2.** *A function $\phi$ is called elementary if it can be written as*

$$\phi(t,\omega) = \sum_j e_j(\omega) 1_{[t_j,t_{j+1})}(t), \quad e_j \in \mathcal{F}_{t_j}.$$

For an elementary function $\phi$ we *define* $\mathcal{I}(\phi)$ as

$$\mathcal{I}(\phi) = \sum_j e_j \left(W_{t_{j+1}\wedge T} - W_{t_j\wedge T}\right) := \sum_j e_j \Delta W_{t_j}.$$

In order to proceed, we need to briefly introduce a conditional expectation and martingales.

**Definition 5.0.3** (Conditional Expectations). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \to \mathbb{R}^d$ be a random variable such that $\mathbb{E}[|X|] < \infty$. If $\mathcal{H} \subset \mathcal{F}$ is a $\sigma$-algebra, then the conditional expectation of $X$ given $\mathcal{H}$, denoted by $\mathbb{E}[X|\mathcal{H}]$, is a random variable such that*

1. *$\mathbb{E}[X|\mathcal{H}]$ is $\mathcal{H}$-measurable.*
2. *$\mathbb{E}[X, A] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}], A]$ for any $A \in \mathcal{H}$.*

*Conditional expectation exists and it is unique a.e. (any two are equal a.e.). Intuitively, the conditional expectation $\mathbb{E}[X|\mathcal{H}]$ is the best guess of $X$ given information $\mathcal{H}$.*

**Properties of Conditional Expectations**

1. $X \to \mathbb{E}[X|\mathcal{H}]$ is linear.
2. Let $\mathcal{H}_1 \subset \mathcal{H}_2$. Then, $\mathbb{E}[\mathbb{E}[X|\mathcal{H}_2]|\mathcal{H}_1] = \mathbb{E}[X|\mathcal{H}_1]$ (Towering property).
3. If $X \in \mathcal{H}$, then $\mathbb{E}[XY|\mathcal{H}] = X\mathbb{E}[Y|\mathcal{H}]$.
4. Convergence Theorems for Conditional Expectations
   (a) Monotone Convergence Theorem
   If $0 \le X_n \le X$ and $X_n \uparrow X$, then $\mathbb{E}[X_n|\mathcal{H}] \uparrow \mathbb{E}[X|\mathcal{H}]$.
   (b) Fatou Theorem
   Let $0 \le X_n$. Then, $\mathbb{E}[\liminf_{n\to\infty} X_n|\mathcal{H}] \le \liminf_{n\to\infty} \mathbb{E}[X_n|\mathcal{H}]$.
   (c) Dominated Convergence Theorem
   Suppose that $X_n \to X$ and $|X_n| \le Z$ with $\mathbb{E}[Z] < \infty$. Then, $\lim_{n\to\infty} \mathbb{E}[X_n|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$.
5. Jensen's Inequality for Conditional Expectation
   Let $\phi$ be convex and $\mathbb{E}[|X|], \mathbb{E}[|\phi(X)|] < \infty$. Then, $\phi(\mathbb{E}[X|\mathcal{H}]) \le \mathbb{E}[\phi(X)|\mathcal{H}]$.

**Definition 5.0.4** (Martingales). *Let $\{\mathcal{F}_t\}$ be a filtration (collection of increasing $\sigma$-algebras). A stochastic process $X = \{X_t\}$ is called a martingale if*

1. $\mathbb{E}[|X_t|] < \infty$ *for each* $t > 0$;
2. $X_t$ *is* $\mathcal{F}_t$ *adapted (*$X_t$ *is* $\mathcal{F}_t$ *measurable);*
3. *for any* $s < t$ $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$.

**Proposition 5.0.5** (Itô Isometry). *Suppose that $\phi$ is bounded and elementary. Then, we have*

$$\mathbb{E}[\mathcal{I}(\phi)^2] = \mathbb{E}[\int_0^T \phi(t,\omega)^2 dt].$$

**Proof.** Note that

$$\phi(t,\omega)^2 = \sum_j e_j(\omega)^2 1_{[t_j,t_{j+1})}(t) \text{ and } \mathbb{E}[\int_0^T \phi(t,\omega)^2 dt] = \sum_j \mathbb{E}[e_j^2](t_{j+1} \wedge T - t_j \wedge T).$$

Also, we have

$$\mathcal{I}(\phi)^2 = \left(\sum_j e_j \Delta W_{t_j}\right)^2 = \sum_j e_j^2 \Delta W_{t_j}^2 + \sum_{j \neq k} e_j e_k \Delta W_{t_j} \Delta W_{t_k}.$$

Hence, the proof will be completed if one can show that

$$\mathbb{E}[e_j e_k \Delta W_{t_j} \Delta W_{t_k}] = 0, \quad j \neq k,$$

and

$$\mathbb{E}[e_j^2 \Delta W_{t_j}^2] = \mathbb{E}[e_j^2](t_{j+1} \wedge T - t_j \wedge T),$$

Using the conditional expectation argument we have for $j < k$

$$\begin{aligned} &\mathbb{E}[e_j e_k \Delta W_{t_j} \Delta W_{t_k}] \\ =&\mathbb{E}\left[\mathbb{E}[e_j e_k \Delta W_{t_j} \Delta W_{t_k} | \mathcal{F}_{t_j}]\right] \\ =&\mathbb{E}\left[e_j e_k \Delta W_{t_j} \mathbb{E}[\Delta W_{t_k} | \mathcal{F}_{t_j}]\right] \\ =&0. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\mathbb{E}[e_j^2 \Delta W_{t_j}^2] \\ =&\mathbb{E}[\mathbb{E}[e_j^2 \Delta W_{t_j}^2 | \mathcal{F}_{t_j}]] \\ =&\mathbb{E}[e_j^2 \mathbb{E}[\left(W_{t_{j+1}\wedge T} - W_{t_j \wedge T}\right)^2 | \mathcal{F}_{t_j}]] \\ =&\mathbb{E}[e_j^2](t_{j+1} \wedge T - t_j \wedge T). \end{aligned}$$

$\square$

Now, we are ready to define the Itô integral $\mathcal{I}(f)$ for $f \in V(T)$.

**Theorem 5.0.6.** *For $f \in V(T)$, one can define*

$$I(f) = \int_0^T f(t,\omega)dW_t$$

*as $L^2$-limit of $\mathcal{I}(\phi_n)$, where $\mathbb{E}[\int_0^T (f(t,\omega) - \phi_n(t,\omega))^2] \to 0$. The Itô integral $\mathcal{I}(f)$ satisfies*

*(1)* $\int_0^T f(t,\omega)dW_t \in \mathcal{F}_T$.

*(2)* $\mathbb{E}[\int_0^T f(t,\omega)dW_t] = 0$.

*(3)* $\mathbb{E}[(\int_0^T f(t,\omega)dW_t)^2] = \mathbb{E}[\int_0^T f(t,\omega)^2 dt]$.

*(4)* $\int_0^S f(t,\omega)dW_t + \int_S^T f(t,\omega)dW_t = \int_0^T f(t,\omega)dW_t$.

*(5)* $\int_0^T af(t,\omega) + bg(t,\omega)dW_t = a\int_0^T f(t,\omega)dW_t + b\int_0^T g(t,\omega)dW_t$.

**Proof.** One of standard ingredients we need is that the elementary functions are dense in $V = V(T)$, which is a standard technique in measure theory. Once, this is established, for $f \in V$ we take a sequence of elementary functions $\{\phi_n\}$ converging to $f$. Note that

$$\mathbb{E}[\left(\int_0^T \phi_n(t,\omega)dW_t - \int_0^T \phi_m(t,\omega)dW_t\right)^2] = \mathbb{E}[\int_0^T (\phi_n(t,\omega) - \phi_m(t,\omega))^2 dt],$$

which shows that $\mathcal{I}(\phi_n)$ is Cauchy in $L^2(\mathbb{P})$, hence it converges in $L^2(\mathbb{P})$. Hence, we can define $\mathcal{I}(f)$ as the $L^2(\mathbb{P})$ limit. It is easy to observe that the limit is independent of the approximating sequence $\phi_n$.
The rest are easy as they holds for elementary functions and the same must hold in the limit. $\square$

Furthermore, one can choose the Itô integral so that $t \to \mathcal{I}(f)(t,\omega)$ is continuous a.s. More precisely, we have

**Theorem 5.0.7.** *For any $T > 0$, the map*

$$t \to \int_0^t f(s,\omega)dW_s$$

*is continuous almost surely for $t \in [0,T]$.*

**Proof.** Two main ingredient for the proof is the Doob's maximal inequality for martingales

$$\mathbb{P}(\sup_{t\leq T}|M_t|\geq\lambda)\leq\frac{1}{\lambda^p}\mathbb{E}[|M_T|^p],\quad p\in[1,\infty),$$

and Borel-Cantelli Lemma.
Let $f\in V$ and choose an approximating sequence $\{\phi_n\}$ of elementary functions converging to $f$ in $L^2([0,T]\otimes\mathbb{P})$. Let

$$\mathcal{I}_n(t,\omega)=\int_0^t\phi_n(s,\omega)dW_s.$$

Then, it is easy to observe that $t\to\mathcal{I}_n(t,\omega)$ is a martingale with respect to $\mathcal{F}_t$: for $s\leq t$

$$\mathbb{E}[\mathcal{I}_n(t,\omega)|\mathcal{F}_s]=\mathcal{I}_n(s,\omega)\quad\text{a.s.}$$

By the Doob's maximal inequality, we have

$$\mathbb{P}\left(\sup_{t\leq T}|\mathcal{I}_n(t,\omega)-\mathcal{I}_m(t,\omega)|>\epsilon\right)\leq\frac{1}{\epsilon^2}\mathbb{E}[|\mathcal{I}_n(T,\omega)-\mathcal{I}_m(T,\omega)|^2].$$

The right-hand side converges to zero as $n,m\to\infty$ and one can choose a subsequence $\{n_k\}$ such that

$$\mathbb{P}\left(\sup_{t\leq T}\left|\mathcal{I}_{n_{k+1}}(t,\omega)-\mathcal{I}_{n_k}(t,\omega)\right|>2^{-k}\right)\leq 2^{-k}.$$

Hence, by Borel-Cantelli Lemma we have

$$\mathbb{P}\left(\limsup_{k}\{\sup_{t\leq T}\left|\mathcal{I}_{n_{k+1}}(t,\omega)-\mathcal{I}_{n_k}(t,\omega)\right|>2^{-k}\}\right)=0,$$

and there exists $\Omega^{'}\subset\Omega$ with $\mathbb{P}(\Omega^{'})=1$ such that for each $\omega\in\Omega^{'}$ there exists $k_1=k_1(\omega)$ such that for all $k\geq k_1$ we have

$$\sup_{t\leq T}\left|\mathcal{I}_{n_{k+1}}(t,\omega)-\mathcal{I}_{n_k}(t,\omega)\right|\leq 2^{-k}.$$

Hence, $\mathcal{I}_{n_k}(t,\omega)$ converges uniformly for all $t\leq T$ and the limit is continuous. Since $L^2$-limit is unique a.s., this establishes the claim. □

## 5.1 Exercises

(1) The purpose of this exercise is to justify the approximation argument in the proof of Theorem 5.0.6.

(a) Let $g\in V$ be bounded and $g(\cdot,\omega)$ is continuous for each $\omega$. Then there exist elementary functions $\phi_n$ such that

$$\mathbb{E}[\int_0^T(g-\phi_n)^2dt]\to 0\text{ as }n\to\infty.$$

*Solution.* Let $\phi_n(t,\omega) = \sum_{j=1}^{n} g(t_j^{(n)},\omega) \cdot 1_{[t_j^{(n)},t_{j+1}^{(n)})}$, where $t_j^{(n)} = \frac{Tj}{n}$. Since $\cdot \to g(\cdot,\omega)$ is continuous, $\phi_n(t,\omega) \to g(t,\omega)$ as $n \to \infty$ and the claim follows from the bounded convergence theorem. □

(b) Let $h \in V$ be bounded. Then there exist bounded functions $g_n \in V$ such that $\cdot \to g_n(\cdot,\omega)$ is continuous and

$$\mathbb{E}[\int_0^T (h-g_n)^2 dt] \to 0 \text{ as } n \to \infty.$$

*Solution.* Let $g_n = \psi_n \star h$, where $\psi_n$ is an approximation to the identity. Then, $g_n$ is smooth in $t$ and $\int_0^T (g_n(s,\omega) - h(s,\omega))^2 ds \to 0$. Now the claim follows from the bounded convergence theorem. □

(c) Let $f \in V$. Then there exists a sequence $h_n \in V$ such that $h_n$ is bounded and

$$\mathbb{E}[\int_0^T (f-h_n)^2 dt] \to 0 \text{ as } n \to \infty.$$

*Solution.* Let $h_n = f \cdot 1_{\{|f| \leq n\}}$. □

# Chapter 6

# Itô Theorem

In this section, we prove the celebrated Itô Formula.

**Definition 6.0.1.** *We say $X = \{X\}_t$ is a Itô process if it can be written as*

$$X_t = X_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dW_s,$$

*where* $\mathbb{P}\left(\int_0^t |u(s,\omega)|\, ds < \infty \text{ for all } t \geq 0\right) = \mathbb{P}\left(\int_0^t |v(s,\omega)|^2\, ds < \infty \text{ for all } t \geq 0\right) = 1$.

We will write it as

$$dX_t = udt + vdW_t.$$

Let $g = g(t,x) \in C^2([0,\infty)\times\mathbb{R})$ and $Y_t = g(t,X_t)$. Here is the statement for the main theorem.

**Theorem 6.0.2.** *Let $X$ be an Itô process given by $dX_t = udt + vdW_t$ and $Y_t = g(t,X_t)$ for $g \in C^2$. Then, we have*

$$dY_t = \frac{\partial g}{\partial t}(t,X_t)dt + \frac{\partial g}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)\cdot(dX_t)^2,$$

*where $(dX_t)^2$ is computed according to the rules*

$$dt\cdot dt = dt\cdot dW_t = 0, \quad dW_t\cdot dW_t = dt.$$

Before we proceed any further, we illustrate some examples that use the Itô Theorem.

**Examples 6.0.3.** *(1)* $\displaystyle\int_0^t W_s dW_s = \frac{1}{2}W_t^2 - \frac{t}{2}$.

*Let $X_t = W_t$ and $Y_t = f(X_t) := \frac{1}{2}W_t^2$. Then, by the Itô Theorem*

$$dY_t = \frac{df}{dx}(Y_t)dX_t + \frac{1}{2}\frac{d^2 f}{dx^2}(Y_t)(dX_t)^2 = W_t dW_t + \frac{1}{2}dt,$$

*or*

$$Y_t - Y_0 = \int_0^t W_s dW_s + \int_0^t \frac{1}{2} ds = \int_0^t W_s dW_s + \frac{t}{2}.$$

*Hence,*

$$\int_0^t W_s dW_s = \frac{1}{2} W_t^2 - \frac{t}{2}.$$

*(2)* $\int_0^t s dW_s = tW_t - \int_0^t W_s ds$*.*

*Let* $X_t = W_t$ *and* $Y_t = f(t, X_t) := tW_t$*. Then, by the Itô Theorem*

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2 = W_t dt + t dW_t,$$

*or*

$$tW_t - 0W_0 = \int_0^t W_s ds + \int_0^t s dW_s \to \int_0^t s dW_s = tW_t - \int_0^t W_s ds.$$

We define a quadratic variation process of $W$ by

$$\langle W, W\rangle(t) := \lim_{\Delta t_k \to 0} \left| W_{t_{k+1}} - W_{t_k} \right|^2,$$

where the convergence is convergence in probability. Recall that $\Delta W_{t_k} = W_{t_{k+1}} - W_{t_k}$.

**Lemma 6.0.4.** *Let* $0 \le t_1 < t_2 < \cdots t_n \le t$ *be a partition of time interval* $[0, t]$*. Then we have*

$$\mathbb{E}\left[\left(\sum_k (\Delta W_{t_k})^2 - t\right)^2\right] = 2\sum_k (\Delta t_k)^2.$$

*Hence,*

$$\langle W, W\rangle(t) \to t \text{ in } L^2(\Omega).$$

**Proof.** First note that

$$\sum_k (\Delta W_{t_k})^2 - t = \sum_k \left((\Delta W_{t_k})^2 - \Delta t_k\right).$$

Hence,

$$\begin{aligned}
\left(\sum_k (\Delta W_{t_k})^2 - t\right)^2 &= \left(\sum_k \left((\Delta W_{t_k})^2 - \Delta t_k\right)\right)^2 \\
&= \sum_k \left((\Delta W_{t_k})^2 - \Delta t_k\right)^2 + 2\sum_{k<j}((\Delta W_{t_k})^2 - \Delta t_k)((\Delta W_{t_j})^2 - \Delta t_j) \\
&= \sum_k \left[(\Delta W_{t_k})^4 - 2\Delta t_k (\Delta W_{t_k})^2 + (\Delta t_k)^2\right] + 2\sum_{k<j}((\Delta W_{t_k})^2 - \Delta t_k)((\Delta W_{t_j})^2 - \Delta t_j).
\end{aligned}$$

Hence it follows from independence of increments and facts that $\mathbb{E}[W_t^2] = t$ and $\mathbb{E}[W_t^4] = 3t^2$

$$\mathbb{E}\left[(\Delta W_{t_k})^4 - 2\Delta t_k(\Delta W_{t_k})^2 + (\Delta t_k)^2\right] = 3(\Delta t_k)^2 - 2(\Delta t_k)^2 + (\Delta t_k)^2 = 2(\Delta t_k)^2$$

For $k < j$, it follows from a conditioning argument

$$\begin{aligned}
&\mathbb{E}[((\Delta W_{t_k})^2 - \Delta t_k)((\Delta W_{t_j})^2 - \Delta t_j)] \\
=&\mathbb{E}\left[\mathbb{E}[((\Delta W_{t_k})^2 - \Delta t_k)((\Delta W_{t_j})^2 - \Delta t_j)|\mathcal{F}_{t_j}]\right] \\
=&\mathbb{E}\left[((\Delta W_{t_k})^2 - \Delta t_k)\mathbb{E}[((\Delta W_{t_j})^2 - \Delta t_j)|\mathcal{F}_{t_j}]\right] \\
=&0.
\end{aligned}$$

This establishes the claim. □

**Proof of Theorem 6.0.2** By an approximation argument, we may assume all functions are bounded. The proof uses the Taylor theorem up to the second order term and a crucial ingredient of the theorem is the quadratic variation of Brownian motions. Note that

$$\begin{aligned}
&Y_t - Y_0 = \sum_j \Delta Y_{t_j} = \sum_j \Delta g(t_j, X_{t_j}) \\
=&\sum_j \frac{\partial g}{\partial t}(t_j, X_{t_j})\Delta t_j + \sum_j \frac{\partial g}{\partial x}(t_j, X_{t_j})\Delta X_{t_j} \\
&+\frac{1}{2}\sum_j \frac{\partial^2 g}{\partial t^2}(t_j, X_{t_j})(\Delta t_j)^2 + \sum_j \frac{\partial^2 g}{\partial t \partial x}(t_j, X_{t_j})\Delta t_j \Delta X_{t_j} + \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial x^2}(t_j, X_{t_j})(\Delta X_{t_j})^2 + \sum_j R_j,
\end{aligned}$$

where $R_j = o(|\Delta t_j|^2 + |\Delta X_j|^2)$.
It is easy to observe that

$$\sum_j \frac{\partial g}{\partial t}(t_j, X_{t_j})\Delta t_j \to \int_0^t \frac{\partial g}{\partial t}(s, X_s)ds, \quad \sum_j \frac{\partial g}{\partial x}(t_j, X_{t_j})\Delta X_{t_j} \to \int_0^t \frac{\partial g}{\partial x}(s, X_s)dX_s,$$

and

$$\sum_j \frac{\partial^2 g}{\partial t^2}(t_j, X_{t_j})(\Delta t_j)^2 \text{ and } \sum_j \frac{\partial^2 g}{\partial t \partial x}(t_j, X_{t_j})\Delta t_j \Delta X_{t_j} \to 0.$$

Note that

$$\begin{aligned}
&\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta X_{t_j})^2 \\
=&\sum_j \frac{\partial^2 g}{\partial x^2}u(t_j,\omega)^2(\Delta t_j)^2 + 2\sum_j \frac{\partial^2 g}{\partial x^2}u(t_j,\omega)v(t_j,\omega)\Delta t_j \Delta W_{t_j} + \sum_j \frac{\partial^2 g}{\partial x^2}v(t_j,\omega)^2(\Delta W_{t_j})^2
\end{aligned}$$

The first two expressions converges to 0 and by Lemma 6.0.4 we have

$$\mathbb{E}\left[\left|\sum_j \frac{\partial^2 g}{\partial x^2}v(t_j,\omega)^2(\Delta W_{t_j})^2 - \sum_j \frac{\partial^2 g}{\partial x^2}v(t_j,\omega)^2\Delta t_j\right|^2\right] \to 0$$

and this shows that

$$\sum_j \frac{\partial^2 g}{\partial x^2} v(t_j,\omega)^2 (\Delta W_{t_j})^2 \to \int_0^t \frac{\partial^2 g}{\partial x^2} v(s,\omega)^2 ds \text{ in } L^2(\mathbb{P}).$$

Finally, we can observe that $R_j \to 0$ by a similar argument and this establishes the claim. $\square$

# Part V

# Stochastic Differential Equations

# Chapter 7

# Examples-Geometric Brownian Motions and OU Processes

In this chapter, we explain two examples that are solutions of SDEs; Geometric Brownian motions and Ornstein-Uhlenbeck Processes.

**Examples 7.0.1** (Geometric Brownian Motions)**.** *Consider a population growth model and let $N = \{N_t\}$, where $N_t$ is a number of certain population. Assume that it satisfies the following equation:*

$$dN_t = aN_t dt + \alpha N_t dW_t.$$

*(1) Find $N_t$. Note that $\frac{dN_t}{N_t} = adt + \alpha dW_t$ and we have*

$$\int_0^t \frac{dN_s}{N_s} = at + \alpha W_t.$$

*We will find $\frac{dN_t}{N_t}$ using the Itô formula. Let $f(x) = \ln x$ and by the Itô formula we have*

$$d(\ln N_t) = \frac{dN_t}{N_t} + \frac{1}{2}(-\frac{1}{N_t^2})(dN_t)^2 = \frac{dN_t}{N_t} - \frac{1}{2N_t^2}\alpha^2 N_t^2 = \frac{dN_t}{N_t} - \frac{1}{2}\alpha^2 dt.$$

*Hence, $d(\ln N_t) = \frac{dN_t}{N_t} - \frac{1}{2}\alpha^2 dt$ and we have*

$$\int_0^t \frac{dN_s}{N_s} = \int_0^t d(\ln N_s) + \frac{1}{2}\alpha^2 ds = \ln \frac{N_t}{N_0} + \frac{1}{2}\alpha^2 t = at + \alpha W_t,$$

*and we have*

$$N_t = N_0 \exp\left((a - \frac{1}{2}\alpha^2)t + \alpha W_t\right).$$

*(2) Assume that the initial population $N_0$ and $W = \{W_t\}$ are independent. Show that*

$$\mathbb{E}[N_t] = \mathbb{E}[N_0]e^{at}.$$

*That is, the expected population is the same as the case without the noise term. By the independence we have*

$$\mathbb{E}[N_t] = \mathbb{E}[N_0]\mathbb{E}[\exp\left((a - \frac{1}{2}\alpha^2)t + \alpha W_t\right)].$$

*We focus on finding* $\mathbb{E}[e^{\alpha W_t}]$*. Let* $Y_t = e^{\alpha W_t}$*. Then, by the Itô formula, we have*

$$dY_t = \alpha Y_t dW_t + \frac{1}{2}\alpha^2 Y_t dt,$$

*and*

$$Y_t = \int_0^t \alpha Y_s dW_s + \frac{1}{2}\alpha^2 \int_0^t Y_s ds.$$

*By taking an expectation and using the fact* $\mathbb{E}[\int_0^t \alpha Y_s dW_s] = 0$ *as it is a martingale, we have*

$$\mathbb{E}[Y_t] = \frac{\alpha^2}{2}\int_0^t \mathbb{E}[Y_s]ds,$$

*and*

$$\frac{d}{dt}\mathbb{E}[Y_t] = \frac{\alpha^2}{2}\mathbb{E}[Y_t].$$

*Hence, we have* $\mathbb{E}[Y_t] = e^{\frac{1}{2}\alpha^2 t}$*.*

**Examples 7.0.2** (Ornstein-Uhlenbeck Processes)**.** *Consider the following Ornstein-Uhlenbeck equation (or Langevin equation), which models Brownian particles under the influence of friction*

$$dX_t = \mu X_t dt + \sigma dW_t, \quad \mu, \sigma \in \mathbb{R}.$$

*(1) Using the variation of parameter* $Y_t = e^{-\mu t}X_t$*, find the solution* $X_t$*.*

*By the stochastic chain rule* $d(A_tB_t) = d(A_t)B_t + A_t d(B_t) + dA_t \cdot dB_t$*, we have*

$$dY_t = -\mu e^{-\mu t}X_t dt + e^{-\mu t}dX_t = e^{-\mu t}\sigma dW_t.$$

*Hence, we have* $Y_t - Y_0 = \int_0^t e^{-\mu s}\sigma dW_s$ *and this gives*

$$X_t = e^{\mu t}X_0 + \sigma\int_0^t e^{\mu(t-s)}dW_s.$$

*(2) Find* $\mathbb{E}[X_t]$ *and Cov*$(X_t, X_s)$*.*

*Clearly,* $\mathbb{E}[X_t] = e^{\mu t}\mathbb{E}[X_0]$ *and by the Itô isometry we have*

$$
\begin{aligned}
\textit{Cov}(X_t, X_s) &= \mathbb{E}\left[(X_t - \mathbb{E}[X_t])(X_s - \mathbb{E}[X_s])\right] \\
=&\mathbb{E}\left[\sigma \int_0^t e^{\mu(t-u)} dW_u \cdot \sigma \int_0^s e^{\mu(s-u)} dW_u\right] \\
=&\sigma^2 e^{\mu(t+s)}\mathbb{E}\left[\int_0^\infty 1_{(0,t)}(u) e^{-\mu u} dW_u \int_0^\infty 1_{(0,s)}(u) e^{-\mu u} dW_u\right] \\
=&\sigma^2 e^{\mu(t+s)} \int_0^{t\wedge s} e^{-2\mu u} du = \sigma^2 e^{\mu(t+s)} \frac{1 - e^{-2\mu(t\wedge s)}}{2\mu} = \frac{\sigma^2}{2\mu}(e^{\mu(t+s)} - e^{\mu|t-s|}).
\end{aligned}
$$

# Chapter 8

# Existence and Uniqueness Theorem of SDE

In this note, we prove the uniqueness and existence of the stochastic differential equations (SDE).

**Theorem 8.0.1.** *Let $T > 0$ and $b(\cdot,\cdot) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma(\cdot,\cdot) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be measurable functions satisfying*

$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|),$$

*and*

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x - y|$$

*for some constants $C$ and $D$. Let $Z$ be a random variable which is independent of the $\sigma$-algebras generated by $B$ and such that*

$$\mathbb{E}[|Z|^2] < \infty.$$

*Then the stochastic differential equation*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad 0 \leq t \leq T, X_0 = Z$$

*has a unique $t$-continuous solution such that $X$ is adapted to the filtration generated by $Z$ and $B$ and $\mathbb{E}[\int_0^T |X_t|^2 dt] < \infty$.*

**Remark 8.0.2.** *The matrix form of the SDE $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$ is given by*

$$\begin{pmatrix} dX_t^{(1)} \\ \vdots \\ dX_t^{(d)} \end{pmatrix} = \begin{pmatrix} b^{(1)}(t, X_t) \\ \vdots \\ b^{(d)}(t, X_t) \end{pmatrix} dt + \begin{pmatrix} \sigma_{11}(t, X_t) & \cdots & \sigma_{1m}(t, X_t) \\ \vdots & \ddots & \vdots \\ \sigma_{d1}(t, X_t) & \cdots & \sigma_{dm}(t, X_t) \end{pmatrix} \begin{pmatrix} dB_t^{(1)} \\ \vdots \\ dB_t^{(m)} \end{pmatrix}.$$

*Alternatively, we can write the SDE as*

$$X_t^{(j)} - X_0^{(j)} = \int_0^t b^{(j)}(s, X_s)ds + \sum_{k=1}^{m} \int_0^t \sigma_{jk}(s, X_s)dB_s^{(k)}, \quad j \in \{1, \cdots, d\}.$$

We need the following Gronwall's Lemma for the proof of Theorem 8.0.1.

**Lemma 8.0.3.** *Let* $u(t), \kappa(t) \geq 0$ *be such that* $u(t) \leq A + \int_0^t v(a)\kappa(a)da$ *for* $0 \leq t \leq T$ *for some constant* $A$*. Then,*

$$u(t) \leq A\exp(\int_0^t \kappa(a)da) \text{ for } 0 \leq t \leq T.$$

**Proof.** Note that we have $\frac{u(s)\kappa(s)}{A + \int_0^s \kappa(a)u(a)da} \leq \kappa(s)$ for $0 \leq s \leq t$. This implies $\frac{d}{ds}\ln\left(A + \int_0^s \kappa(a)u(a)da\right)$ $\kappa(s)$. By integrating from 0 to $t$, we have $\ln(A + \int_0^t \kappa(a)u(a))da - \ln A \leq \int_0^t \kappa(a)da$ and this implies

$$A + \int_0^t \kappa(a)u(a)da \leq A\exp(\int_0^t \kappa(a)da).$$

Finally,

$$u(t) \leq A + \int_0^t \kappa(a)u(a)da \leq A\exp(\int_0^t \kappa(a)da).$$

□

**Proof of Theorem 8.0.1 Uniqueness** The main tool to prove the uniqueness is the Gronwall's lemma. Let $X_t$ and $\tilde{X}_t$ be two solutions of the following SDEs

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, X_0 = Y_0, t \in [0, T],$$

and

$$d\tilde{X}_t = b(t, \tilde{X}_t)dt + \sigma(t, \tilde{X}_t)dB_t, \tilde{X}_0 = \tilde{Y}_0, t \in [0, T],$$

Using a simple inequality $(x + y + z)^2 \leq 3(x^2 + y^2 + z^2)$ we have

$$\begin{aligned}
&\mathbb{E}[|X_t - \tilde{X}_t|^2] \\
=&\mathbb{E}\left[\left((Y_0 - \tilde{Y}_0) + \int_0^t (b(s, X_s) - b(s, \tilde{X}_s))ds + \int_0^t (\sigma(s, X_s) - \sigma(s, \tilde{X}_s))dB_s\right)^2\right] \\
\leq&3\left(\mathbb{E}[(Y_0 - \tilde{Y}_0)^2] + (\int_0^t (b(s, X_s) - b(s, \tilde{X}_s))ds)^2 + (\int_0^t (\sigma(s, X_s) - \sigma(s, \tilde{X}_s))dB_s)^2\right).
\end{aligned}$$

By Cauchy-Schwarz inequality and the condition on $b$, we have

$$\begin{aligned}&(\int_0^t (b(s,X_s)-b(s,\tilde{X}_s))ds)^2\\ \leq&\int_0^t 1ds\cdot\int_0^t (b(s,X_s)-b(s,\tilde{X}_s))^2ds\\ \leq& tD^2\int_0^t |X_s-\tilde{X}_s|^2ds.\end{aligned}$$

By Itô isometry and the condition on $b$, we have

$$\begin{aligned}&\mathbb{E}[(\int_0^t (\sigma(s,X_s)-\sigma(s,\tilde{X}_s))dB_s)^2]\\ =&\mathbb{E}[\int_0^t (\sigma(s,X_s)-\sigma(s,\tilde{X}_s))^2ds]\\ \leq& D^2\mathbb{E}[\int_0^t |X_s-\tilde{X}_s|^2].\end{aligned}$$

Hence, we conclude that

$$\mathbb{E}[|X_t-\tilde{X}_t|^2]\leq 3\mathbb{E}[(Y_0-\tilde{Y}_0)^2]+3D^2(1+T)\mathbb{E}[\int_0^t |X_s-\tilde{X}_s|^2].$$

By Gronwall's lemma we conclude that

$$\mathbb{E}[|X_t-\tilde{X}_t|^2]\leq 3\mathbb{E}[(Y_0-\tilde{Y}_0)^2]\exp(3D^2(1+T)t),\quad t\in[0,T].$$

As $Y_0=\tilde{Y}_0=Z$, we conclude that $\mathbb{E}[|X_t-\tilde{X}_t|^2]=0$ for all $t\in[0,T]$ and $X_t=\tilde{X}_t$ a.e. for all $t\in[0,T]$. Hence, there is $\Omega'\subset\Omega$ with $\mathbb{P}(\Omega')=1$ such that

$$X_t(\omega)=\tilde{X}_t(\omega)\text{ for all }t\in[0,T]\cap\mathbb{Q}\text{ and }\omega\in\Omega'.$$

From the continuity, we conclude that

$$X_t(\omega)=\tilde{X}_t(\omega)\text{ for all }t\in[0,T]\text{ and }\omega\in\Omega'.$$

**Existence and Continuity**
We define $Y_t^{(n)}$ inductively as

$$Y_t^{(0)}=X_0,$$

and

$$Y_t^{(n+1)}=X_0+\int_0^t b(s,Y_s^{(n)})ds+\int_0^t\sigma(s,Y_s^{(n)})dB_s.$$

By a similar calculation as above, we have

$$\mathbb{E}[|Y_t^{(n+1)}-Y_t^{(n)}|^2]\leq 3D^2(1+t)\mathbb{E}[\int_0^t |Y_s^{(n)}-Y_s^{(n-1)}|^2ds]\tag{8.1}$$

We also have

$$\begin{aligned}
&\mathbb{E}[|Y_t^{(1)} - Y_t^{(0)}|^2] \\
=&\mathbb{E}[\Big(\int_0^t b(s, X_0)ds + \int_0^t \sigma(s, X_0)dB_s\Big)^2] \\
\leq& 2\mathbb{E}\left[(\int_0^t b(s, X_0)ds)^2 + (\int_0^t \sigma(s, X_0)dB_s)^2\right] \\
\leq& 2t\mathbb{E}[\int_0^t b(s, X_0)^2 ds] + 2\mathbb{E}[\int_0^t \sigma(s, X_0)^2 ds] \\
\leq& 2t(1+t)C^2\mathbb{E}[(1+X_0)^2].
\end{aligned} \tag{8.2}$$

From (8.1) and (8.2) we conclude that

$$\mathbb{E}[|Y_t^{(n+1)} - Y_t^{(n)}|^2] \leq \frac{c^{n+1}t^{n+1}}{(n+1)!}.$$

This shows that $Y_t^{(n)}$ is uniformly Caychy in $L^2(\Omega)$ and it converges uniformly to $X_t$, $t \in [0, T]$. As all $Y_t^{(n)}$ are continuous, this shows that $X_t$ is continuous as a uniform limit of continuous functions.
Finally, note that by Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t b(s, Y_s^{(n)}) - b(s, X_s)ds\right)^2\right] \\
\leq& \mathbb{E}[t\int_0^t (b(s, Y_s^{(n)}) - b(s, X_s))^2 ds] \\
\leq& tD^2\mathbb{E}[\int_0^t |Y_s^{(n)} - X_s|^2 ds] \to 0 \text{ as } n \to \infty.
\end{aligned}$$

by a uniform convergence theorem. By a similar argument with Itô isometry, we also have

$$\mathbb{E}\left[\left(\int_0^t \sigma(s, Y_s^{(n)}) - \sigma(s, X_s)dB_s\right)^2\right] \to 0 \text{ as } n \to \infty.$$

Hence, by taking a limit to

$$Y_t^{(n+1)} = X_0 + \int_0^t b(s, Y_s^{(n)})ds + \int_0^t \sigma(s, Y_s^{(n)})dB_s,$$

we conclude that $X_t$ satisfies the SDE.

□

# Part VI

# Bibliography

# Bibliography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**Hyunchul Park**
Department of Mathematics, State University of New York at New Paltz, NY 12561, USA
E-mail: `parkh@newpaltz.edu`